YISHUN INNOVA JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION
**Higher 2**

CANDIDATE NAME

CG

INDEX NO

# MATHEMATICS 9758/01

**24 August 2022**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

**3 hours**

## READ THESE INSTRUCTIONS FIRST

Write your CG, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

**For Examiners' Use**

| Question | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Marks | | | | | | |

| Question | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|
| Marks | | | | | |

| Total marks | /100 |
|---|---|

This document consists of **19** printed pages and **1** blank page.

**1** **Do not use a calculator in answering this question.**

It is given that $2-\mathrm{i}$ is a root of the equation $2z^3+az^2-2z+b=0$.

**(a)** Find the values of the real numbers $a$ and $b$ and the remaining roots of the equation. [4]

**(b)** Using these values of $a$ and $b$, deduce the roots of the equation

$$bz^3-2z^2+az+2=0.$$ [2]

**2** Figure 1 shows a sketch of the curve $y = \mathrm{f}(x)$ and $A$ is the region under the curve between $x = 1$ and $x = 3$. Yvonne and Irene use different ways to draw 4 rectangles of equal width, $h$, to estimate the area of $A$.

Figure 2 shows 4 rectangles drawn by Yvonne, with the curve intersecting each rectangle at the mid-point of its width. The $x$-coordinates of the mid-points are $m_1$, $m_2$, $m_3$ and $m_4$.

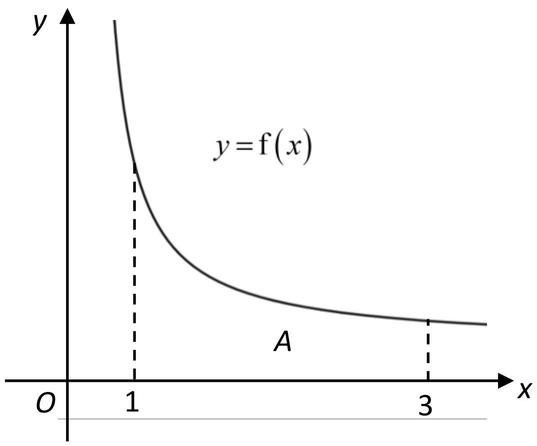



**Figure 1**

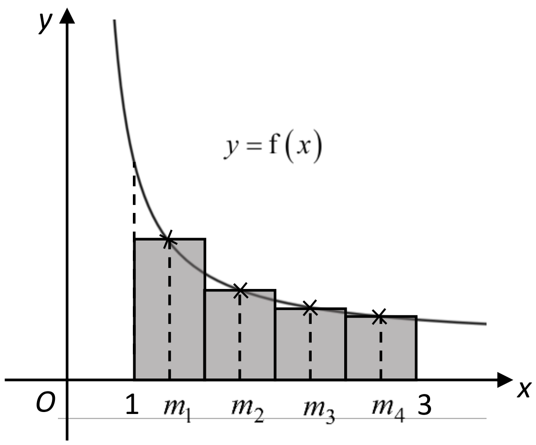



**Figure 2**

**(a)** State the values of $h$, $m_1$, $m_2$, $m_3$ and $m_4$. [2]

**2 [Continued]**

**(b)** The sum of area of rectangles using Yvonne's method is denoted by $B$. Show that $B = h\sum_{r=0}^{3}\left(\mathrm{f}\left(a+rh\right)\right)$, where the value of $a$ is to be determined. [2]

**(c)** Irene finds that the sum of area of 4 rectangles that she has drawn is $C = h\sum_{r=0}^{3}\left(\mathrm{f}\left(1+rh\right)\right)$. Draw these rectangles in **Figure 1**. [1]

You are now given that $\mathrm{f}\left(x\right)=\dfrac{1}{x}+1$.

**(d)** By finding the numerical values of $B$, $C$ and the actual area of region $A$, explain how these values **<u>and</u>** the rectangles drawn in Figures 1 and 2 show that Yvonne's estimation of the area of $A$ is better than Irene's. [2]

**3** **(a)** On the same axes, sketch the graphs of $y=\left|x(x-5)\right|$ and $y=\sqrt{2}\left|x-5\right|$. [2]

**(b)** Hence, or otherwise, solve exactly the inequality $\left|x(x-5)\right|>\sqrt{2}\left|x-5\right|$. [4]

**4** The points $P$, $Q$ and $R$ have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively, where $\mathbf{p}$ and $\mathbf{q}$ are non-zero and non-parallel. The points $P$ and $Q$ are fixed and $R$ varies.

**(a)** Given that $\mathbf{r}\times\mathbf{q}=-\mathbf{q}\times\mathbf{p}$, describe geometrically the set of all possible positions of the point $R$. [4]

**(b)** Given instead that $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{p}=\begin{pmatrix}1\\-2\\3\end{pmatrix}$, $\mathbf{q}=\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}$, and that $\mathbf{q}\cdot(\mathbf{p}-\mathbf{r})=0$, find the relationship between $x$, $y$, and $z$ in terms of $q_1$, $q_2$ and $q_3$. Describe the set of all possible positions of the point $R$ in this case. [3]

**(c)** It is now given that $|\mathbf{q}|=1$ and $C$ is a point with position vector $\mathbf{c}$ such that $\mathbf{q}\cdot(\mathbf{p}-\mathbf{c})\neq 0$ . Give a geometrical meaning of $|\mathbf{q}\cdot(\mathbf{p}-\mathbf{c})|$. [1]

**5** **(a)** Show that $\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}=\frac{\mathrm{f}(r)}{(r+1)!}$ , where $\mathrm{f}(r)$ is a function in $r$ to be found. [1]

The sum $\sum_{r=2}^{N}\frac{\mathrm{f}(r)}{(r+1)!}$ is denoted by $S_N$ .

**(b)** Using your answer in part **(a)**, find $S_N$ in terms of $N$ . [3]

**5 [Continued]**

**(c)** Give a reason why $S_N$ converges and find the exact value of $S_\infty$. [2]

**(d)** Find the smallest value of $N$ such that $S_N$ is within $10^{-7}$ of $S_\infty$. [2]

**6** **(a)** Show that $1+\mathrm{e}^{-\mathrm{i}\alpha}=2\cos\dfrac{\alpha}{2}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}$, where $-\pi<\alpha\leq\pi$. [2]

**(b)** Hence or otherwise, show that

$$\left(1+\mathrm{e}^{-\mathrm{i}\alpha}\right)^3-\left(1+\mathrm{e}^{\mathrm{i}\alpha}\right)^3=-16\mathrm{i}\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right).$$ [3]

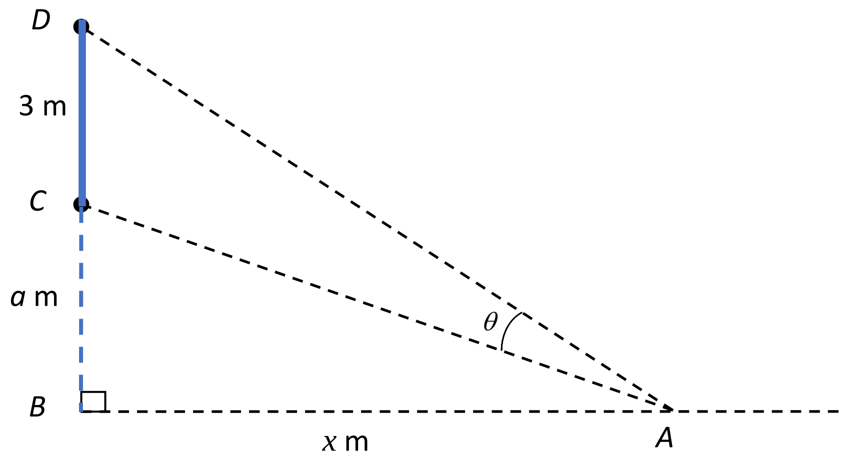
**6 [Continued]**

**(c)** Given further that $0 < \alpha < \frac{2}{3}\pi$ and $z = \left(1 + \mathrm{e}^{-\mathrm{i}\alpha}\right)^3 - \left(1 + \mathrm{e}^{\mathrm{i}\alpha}\right)^3$, deduce the modulus and argument of $z$. Express your answers in terms of $\alpha$ whenever applicable. [2]

**7** **(a)** It is given that $\mathrm{f}(x)=\ln(1+\sin 2x)+2$, where $0 \le x \le \dfrac{\pi}{2}$. By using differentiation, find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in $x^2$. [5]

**(b)** Given that $x$ is a sufficiently small angle, find the series expansion of $\dfrac{1}{\cos 2x+\sin x}$, up to and including the term in $x^2$. [4]

**8**




The diagram shows a cross-sectional view of an advertisement sign *CD* hung against a vertical wall, where the point *C* is *a* metres above the eye level *AB* of an observer who is *x* metres from *B*. The distance *CD* is 3 metres and angle *CAD* is $\theta$.

**(a)** By expressing $\theta$ as the difference of two angles, or otherwise, show that

$$\tan\theta = \frac{3x}{x^2 + 3a + a^2} .$$ [3]

**(b)** Find, in terms of $a$, the value of $x$ which maximises $\tan\theta$, simplifying your answer. Find also the corresponding value of $\tan\theta$. You do not need to show that $\tan\theta$ is maximum. [4]

**8 [Continued]**

**(c)** Find $\tan ADB$ when $\tan\theta$ is maximum, expressing your answer in terms of $a$. Find the approximate value of angle $ADB$ when $a$ is much greater than 3. [3]

**9** A curve $C$ has parametric equations

$$x = a\cos 2t, \qquad y = 2a\cos t,$$

for $0 \le t \le \frac{\pi}{2}$, where $a$ is a positive constant.

**(a)** Show that the equation of normal to the curve at the point $P\left(a\cos 2p,\ 2a\cos p\right)$ is $y = -2\cos p\left(x - 2a\cos^2 p\right)$. [3]

**(b)** The normal at $P$ meets the $x$-axis at the point $R$. Show that the area enclosed by the $x$-axis, the normal at $P$ and $C$ is given by

$$4a^2 \int_{t_1}^{t_2} \cos t \sin 2t \, \mathrm{d}t + a^2 \cos p,$$

where the values of $t_1$ and $t_2$ should be stated. [6]

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**(c)** Hence find in terms of $a$, the exact area in part **(b)** given now that $p = \dfrac{\pi}{3}$. [3]

**10** Alan and Betty bought an apartment at \$450, 000. They are eligible to take a housing loan, up to 85% of the cost of the apartment, for a maximum of 30 years.

After careful consideration, the couple decides to borrow 85% of the cost of the apartment. They will make a cash repayment of \$$x$ at the beginning of each month, starting 1st July 2022. Interest will be charged with effect from 31st July 2022 at a monthly interest rate of 0.2% for the remaining amount owed at the end of each month.

**(a)** Find the amount of money owed on 31st of July 2022 after the interest for the month has been added. Express your answer in terms of $x$. [1]

**(b)** Show that the total amount of money owed after the $n$th repayment at the beginning of the month is

$$1.002^{n-1}(382500) - 500x(1.002^{n} - 1).$$ [4]

**(c)** Find the earliest date on which the couple will be able to pay off the loan completely if $x = 2000$, and state the amount of repayment on this date. [4]

**(d)** If the couple wishes to pay off the loan completely on 1st Jan 2050 (after the repayment on this day), what should the monthly repayment be? [3]

**11** Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time $t$ years is denoted by $N$, where $N$ is treated as a continuous variable. It is given that the rate of increase of $N$ with respect to $t$ is proportional to $(N-120)$.

**(a)** Write down a differential equation relating $N$ and $t$. [1]

Initially, the number of plants was 600. It is noted that at a time when there were 750 plants, the number of plants was increasing at a rate of 63 per year.

**(b)** Express $N$ in terms of $t$. [6]

**(c)** The naturalist has a target of increasing the number of plants from 600 to 2500 within 15 years. Justify whether this target will be met. [2]

Alongside the monitoring of the number of plants of this rare species, naturalists also study the rate of increase of its height, $h$ cm, with respect to time, $t$ years after planting. The height of a plant is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{2}\sqrt{\left(24 - \frac{1}{3}h\right)}.$$

The plant is planted as a seedling of negligible height, so that $h = 0$ when $t = 0$.

**(d)** State the maximum height of the plant, according to this model. [1]

**(e)** Find an expression for $t$ in terms of $h$, and hence find the time the plant takes to reach 24 cm. [5]

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YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
**Higher 2**

CANDIDATE NAME

CG                                   INDEX NO

# MATHEMATICS 9758/02

**15 September 2022**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

**3 hours**

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| Question | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Marks | | | | |
| Question | 5 | 6 | 7 | 8 |
| Marks | | | | |
| Question | 9 | 10 | 11 | 12 |
| Marks | | | | |

| Total marks | / 100 |
|---|---|

This document consists of **23** printed pages and **1** blank page.

## Section A: Pure Mathematics [40 marks]

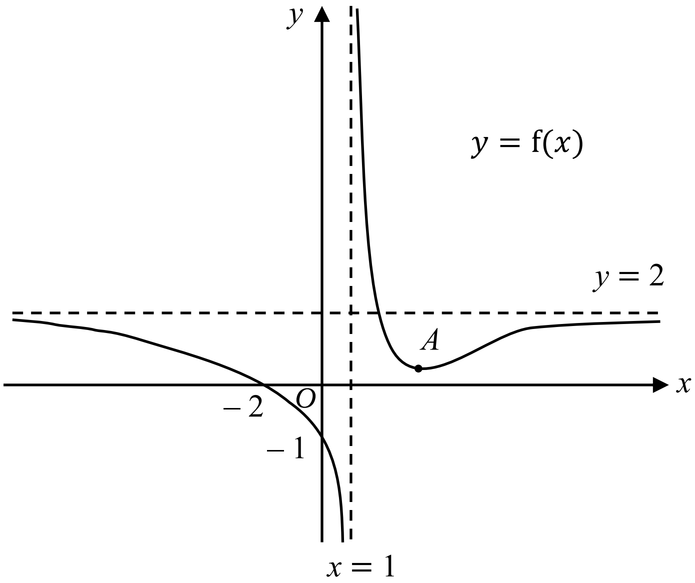
**1** It is given that $I = \int \frac{x}{\sqrt{4-2x}} \, \mathrm{d}x$.

**(a)** Use integration by parts to find an expression for *I*. [2]

**(b)** Use the substitution $u = 4 - 2x$ to find another expression for *I*. [2]

**(c)** Show algebraically that the answers to parts **(a)** and **(b)** differ by a constant. [2]

**2**




The diagram shows the curve $y = \mathrm{f}(x)$ with a turning point $A\left(3, \frac{1}{2}\right)$. The curve crosses the axes at $x = -2$ and $y = -1$ and the lines $x = 1$ and $y = 2$ are the asymptotes of the curve.

Sketch the following curves on separate diagrams, stating, if it is possible to do so, the equations of any asymptotes and the coordinates of any points where each curve crosses the axes and of any turning points.

**(a)** $y = \mathrm{f}'(x)$, [2]

**(b)** $y = \dfrac{1}{\mathrm{f}\,(x)}$. [3]

**3** Functions f and g are defined by

$$\mathrm{f} : x \mapsto x^2 + 3x - 1,\ x \in \mathbb{R},\ x \le k,$$

$$\mathrm{g} : x \mapsto \sqrt{x+5},\ x \in \mathbb{R},\ x \ge -5.$$

**(a)** Given that $\mathrm{f}^{-1}$ exists, state the largest possible value of $k$. Using this value of $k$, find $\mathrm{f}^{-1}(x)$. [3]

For the rest of this question, let $k = -2$.

**(b)** Find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

**(c)** Determine whether the composite functions fg and gf exist. If the composite function exists, give a definition (including the domain) of the function. [3]

**(d)** Hence find the exact range of the composite function that exists. [1]

**4** **(a)** State a sequence of transformations that will transform the curve with equation $y=\ln x$ onto the curve with equation $y=\ln\left(2x+3\right)^3$. [3]

A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x)=\begin{cases}\ln 64 & \text{for } x>\dfrac{1}{2},\\ \ln\left(2x+3\right)^3 & \text{for } -\dfrac{1}{2}\le x\le\dfrac{1}{2},\\ \ln 8 & \text{otherwise.}\end{cases}$$

**(b)** Sketch the curve for $-1 \le x \le 1$. [3]

**(c)** Find the numerical value of the volume generated when the region bounded by the curve $y = \mathrm{f}(x)$, the line $x = 1$ and the line $y = \ln 27$ is rotated completely about the $y$-axis. Give your answer correct to 3 decimal places. [3]

**5** The plane $p$ has equation $\mathbf{r}=\begin{pmatrix}6\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$, where $\lambda$ and $\mu$ are parameters.

The line $l$ passes through the points $A$ and $B$ with position vectors $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $4\mathbf{j}+2\mathbf{k}$ respectively.

**(a)** Find the coordinates of the point of intersection between $p$ and $l$. [5]

**(b)** Find the cartesian equations of the planes such that the perpendicular distance from each plane to $p$ is $\sqrt{41}$. [3]

Another line $m$ has equation $\dfrac{1-x}{3} = y+2 = \dfrac{z-3}{a}$.

**(c)** Find the value of $a$ such that $p$ and $m$ do not meet in a unique point. [3]

## Section B: Probability and Statistics [60 marks]

**6** A group of 12 people consists of 5 men and 7 women. One of the women is the wife of one of the men.

**(a)** How many committees of 5 can be formed which include at least 3 women? [2]

**(b)** The 12 people sit at random at a round table. Find the probability that the husband and his wife are seated together and no two men are next to each other. [3]

**7** Bag $A$ contains 4 balls numbered 3, 5, 6 and 9. Bag $B$ contains 5 balls numbered 1, 2, 7, 9 and 9. Bag $C$ contains 8 balls numbered 3, 4, 4, 8, 8, 9, 9 and 9. All the balls are indistinguishable apart from the number on the balls. One ball is selected at random from each bag.

- $X$ is the event that exactly two of the selected balls have the same number.
- $Y$ is the event that the ball selected from bag $A$ is numbered 5.

**(a)** Show that $\mathrm{P}(X)=\dfrac{21}{80}$. [2]

**(b)** Find $\mathrm{P}(X \cap Y)$ and hence determine whether $X$ and $Y$ are independent. [3]

**7** **[Continued]**

**(c)** Find the probability that one ball is numbered 7, given that exactly two of the selected balls have the same number. [2]

**8** A fair cubical die has faces numbered with 4 distinct numbers $a$, $2a$, $b$ or $3b$, with some faces bearing the same number, where $a \neq b$. The die is thrown once and the number on the uppermost face is the score $T$.

It is given that the mode of $T$ is $a$ and that $\mathrm{P}(T = 2a) = \mathrm{P}(T = 3b) = \frac{1}{6}$.

**(a)** Draw a table to show the probability distribution of $T$. [2]

**(b)** Given that the mean score is $\frac{25}{6}$, find the variance of the score in terms of $a$. [5]

**9** A flower shop makes 75 bouquets of flowers daily. On average, $p$% of the bouquets have LED lights. Assume that $X$, the number of bouquets of flowers with LED lights made daily, follows a binomial distribution.

**(a)** Given that there is a probability of 0.0288 that fewer than 2 bouquets made in a day have LED lights, write down an equation in terms of $p$ and hence find $p$ correct to 4 decimal places. [3]

It is now given that $p = 7.5$.

**(b)** Find the most likely number of bouquets with LED lights made in a day. [2]

**(c)** 30 days are randomly selected. Find the probability that the mean number of bouquets with LED lights made per day is at least 5. [3]

**10** In an experiment, a chemist applied different quantities, $x$ ml, of a chemical to 7 samples of a type of metal, and the times, $t$ hours, for the metal to discolour were measured. The results are given in the table.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 1.2 | 2.0 | 2.7 | 3.8 | 4.8 | 5.6 | 7.0 |
| $t$ | 2.2 | 4.5 | 5.8 | 7.3 | 8.0 | 9.0 | 10.5 |

**(a)** Draw a scatter diagram for these values, labelling the axes. [1]

**(b)** Find, correct to 4 decimal places, the product moment correlation coefficient between

**(i)** $\ln x$ and $t$,

**(ii)** $e^{-x}$ and $t$. [2]

**(c)** Explain which of the two cases in part **(b)** is more appropriate and find the equation of a suitable regression line for this case. [3]

**(d)** Use the equation of your regression line to estimate the value of the quantity of chemical applied to the metal when the time taken for the metal to discolour is 8.5 hours. Explain whether your estimate is reliable. [2]

**11** Farm $A$ claims that the duck eggs from their farm have a mean mass of 70 grams. A random sample of 50 duck eggs is selected. The masses, $x$ grams, are summarised as follows.

$$\sum(x-70)=186.35, \quad \sum(x-70)^2=10494.$$

**(a)** Calculate unbiased estimates of the population mean and variance. [2]

**(b)** Test, at the 5% level of significance, whether Farm $A$'s claim is valid. [4]

**(c)** State, with a reason, whether it is necessary to assume a normal distribution for the test to be valid. [1]

**(d)** Explain the meaning of ‘at the 5% level of significance’ in the context of the question. [1]

Farm *B* claims that their duck eggs have a mean mass of more than 70 grams. A random sample of 40 duck eggs is taken, and it is found that their mean mass and variance are $k$ grams and 146 grams$^2$ respectively. Given that a test at the 3% significance level indicates that Farm *B*’s claim is valid, find the set of values of $k$. [4]

**12** **In this question you should state clearly the values of the parameters of any normal distribution you use.**

A supermarket sells two types of sugar. White sugar is sold in packets with the labelled mass of 1 kg. The mass of a packet of white sugar may be regarded as a normally distributed random variable with mean 1.05 kg and standard deviation 0.03 kg.

**(a)** What mass is exceeded by 80% of the packets of white sugar? Give your answer correct to 3 decimal places. [1]

**(b)** A packet of white sugar that weighs less than 98% of the labelled mass is considered underweight. Find the probability that at most 1 out of 10 randomly chosen packets of white sugar is underweight. [3]

The masses of packets of brown sugar are normally distributed with mean $m$ kg and standard deviation 0.05 kg and the masses of packets of white sugar and brown sugar have independent normal distributions.

**(c)** Given that the probability that the total mass of 4 randomly chosen packets of white sugar exceeds twice the mass of a randomly chosen packet of brown sugar is 0.15, find $m$. [5]

It is now given that $m = 2.03$.

**(d)** Find the probability that the average mass of 4 randomly chosen packets of white sugar and 5 randomly chosen packets of brown sugar exceeds 1.6 kg. [4]

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