VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2022

CANDIDATE NAME	

CLASS		INDEX NUMBER	

H2 MATHEMATICS **9758/01**
Paper 1 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)
Writing paper

READ THESE INSTRUCTIONS FIRST

Write your class and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

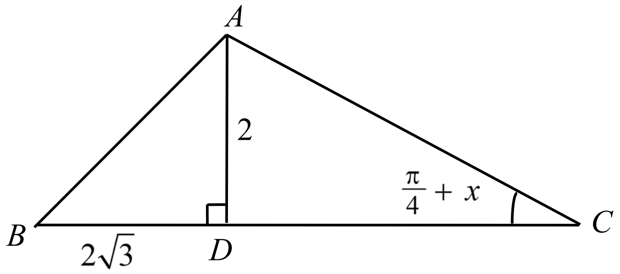
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of **21** printed pages and **3** blank pages.

1 A curve has equation $2x + y + 2 = (x + y)^2 + \dfrac{x^2}{1 + x^2}$.

Find the equation of the normal to the curve at the point $(0, 2)$. [5]

2 The diagram shows triangle ABC, where angle $ACD = \left(\frac{\pi}{4} + x\right)$ radians. Point D is on BC such that $AD = 2$ and $BD = 2\sqrt{3}$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC \approx k(1 + \sqrt{3} - 2x + 2x^2),$$

where k is a constant to be determined. [5]

3 **(a)** Find $\int (\ln x)^2 \, \mathrm{d}x$. [3]

(b) Find $\int \frac{(\sin x + \cos x)^2}{\cos(2x) - 2x} \, \mathrm{d}x$. [3]

4 A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} 2 - |x+1| & \text{for } -3 < x \leqslant 1, \\ 2 - 2(x-2)^2 & \text{for } 1 \leqslant x < 2. \end{cases}$$

(i) Sketch the curve for $-3 < x < 2$. [3]

(ii) Hence, solve the inequality $\mathrm{f}(x) \leqslant 0.1(x-1)^2$ for $-3 < x < 2$, leaving your answers in an exact form. [4]

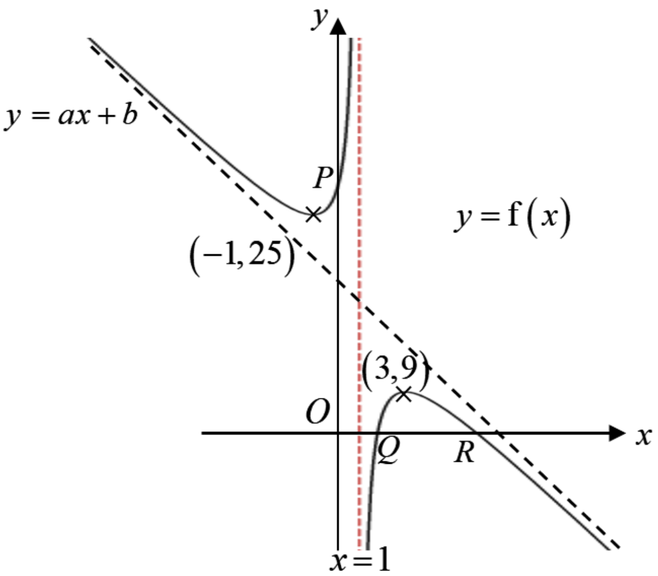
5 **Do not use a calculator in answering this question.**
The complex number z satisfies the equation

$$z^2 - (4 + \mathrm{i})z + 2(\mathrm{i} - t) = 0,$$

where t is a real number. It is given that one root is of the form $k - k\mathrm{i}$, where k is real and positive. Find t and k, and the other root of the equation. [7]

[continued]

6



It is given that $f(x) = ax + b + \dfrac{c}{x-1}$, where a, b and c are constants. The diagram shows the curve with equation $y = f(x)$. The curve crosses the axes at points P, Q and R, and has stationary points at $(-1,\ 25)$ and $(3,\ 9)$.

Find the values of the constants a, b and c. [4]

It is now given that points P, Q and R have coordinates $(0, 27)$, $\left(\frac{3}{2}, 0\right)$ and $(9, 0)$ respectively.

Sketch the curve

(i) $y = \mathrm{f}(|x|)$, [2]

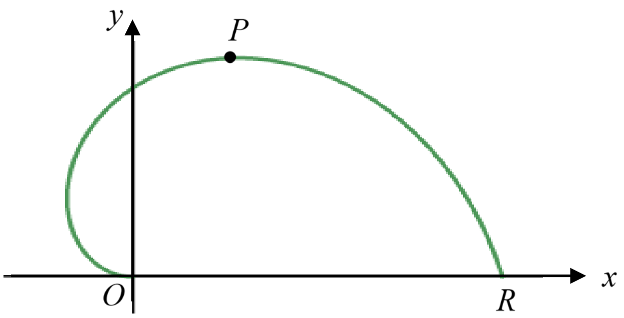
(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes, the coordinates of any points where the curve crosses the axes and of any turning point(s).

7 The diagram below shows the curve C with parametric equations given by

$$x = -3\theta\cos 3\theta, \quad y = 4\theta\sin 3\theta, \quad \text{for } 0 \leqslant \theta \leqslant \frac{\pi}{3}.$$

Point P lies on C with parameter θ and C crosses the x-axis at the origin O and the point R.



(a) Find the area of the region bounded by C and the line $y = \frac{2\pi}{3} - \frac{2x}{3}$, giving your answer correct to 2 decimal places. [4]

(b) Use differentiation to find the maximum value of the area of triangle OPR as θ varies, proving that it is a maximum. [6]

8 **(a)** The sum of the first n terms of a sequence is given by $S_n = \dfrac{n^2 + 5n}{8}$. Show that the sequence follows an arithmetic progression with common difference d, where d is to be determined.

In a geometric progression, the first term is 100 and its common ratio is $3d$. Find the smallest value of k such that the sum of the first k terms of the arithmetic progression is greater than the sum of the first 30 terms of the geometric progression. [6]

(b) The first and second terms of a geometric sequence are $u_1 = a$ and $u_2 = a^2 - a$. If all the terms of the sequence are positive, find the set of values of a for which $\sum_{r=1}^{\infty} u_r$ converges. [2]

For this sequence, it is known that the sum of all the terms after the nth term is equal to the nth term. Find the value of a and hence the value of $\sum_{r=1}^{\infty} u_r$. [3]

9 The curve C has equation $\frac{1}{3}x^2 + y^2 - 2y = 0$.

(i) Sketch C. [2]

(ii) Use the substitution $x = p\sin\theta$ to show that

$$\int_0^p \sqrt{p^2 - x^2}\, dx = \frac{p^2\pi}{4},$$

where p is a positive constant. [4]

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(iii) The region R is bounded by C, the line $x = \sqrt{3}$ and the x-axis. Find the exact area of R. [3]

(iv) R is rotated completely about the y-axis. Find the exact volume of the solid obtained. [3]

(v) Describe a pair of transformations which transforms the graph of C onto the graph of $x^2 + y^2 = 1$. [2]

10 Workers are installing zip lines at an adventure park. The points (x, y, z) are defined relative to the entrance at $(0,0,0)$ on ground level, where units are in metres. The ticketing booth at $(100,100,1)$ and lockers at $(200,120,0)$ are also on ground level. Zip lines are laid in straight lines and the widths of zip lines can be neglected. The ground level of the park is modelled as a plane.

(i) Find a cartesian equation of the plane that models the ground level of the park. [2]

A zip line connects the points $P(300,120,30)$ and $Q(300,320,25)$, and is modelled as a segment of the line l. The façade of a building nearby can be modelled as part of the plane with equation $\mathbf{r}.\begin{pmatrix} 1 \\ -5 \\ 100 \end{pmatrix} = 0$. As a safety requirement, every point on the zip line must be at a distance of at least 10 metres away from the façade of the building.

(ii) Write down a vector equation of l. Hence, or otherwise, determine if the zip line passes the safety requirement. [4]

The workers need to install another zip line from Q to $R(127, 220, a)$, where $0 < a < 30$, and the angle PQR is given to be $60°$.

(iii) Find the value of a, leaving your answer to 3 decimal places. [3]

10 [Continued]

The façade of the building meets the ground level of the park at line m. A worker sets up a transmitter at point S on line m such that S is nearest to Q.

(iv) Find a vector equation of m and the distance from Q to S. [4]

11 A tank contains 500 litres of water in which 100 g of a poisonous chemical called Prokrastenate is dissolved. A solution containing 0.1 g of Prokrastenate per litre is pumped into the tank at a rate of 5 litres per minute, and the well-mixed solution is pumped out at the same rate. By letting x grams be the amount of Prokrastenate in the tank after t minutes, show that

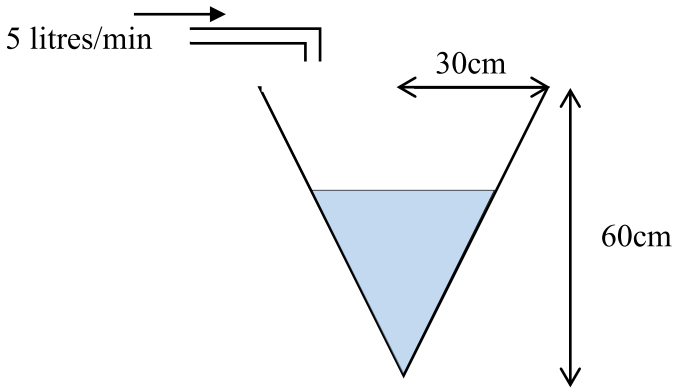
$$\frac{dx}{dt} = \frac{k - x}{100},$$

where k is a constant to be determined. [2]

Find x in terms of t and find the value of t when $x = 75$. [5]

11 [Continued]

The well-mixed solution that is pumped out flows into an empty container, in the form of an open inverted cone with a height of 60 cm and base radius 30 cm, at the same rate (see diagram).



Given that 1 litre = 1000 cm^3,

(i) Show that the volume of the well-mixed solution in the container, V cm^3 can be expressed as $V = \dfrac{\pi h^3}{12}$, where h cm is the depth of the solution at that instant. [2]

(ii) Hence, or otherwise, find the rate of change of the depth of solution after 5 minutes. [4]

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]

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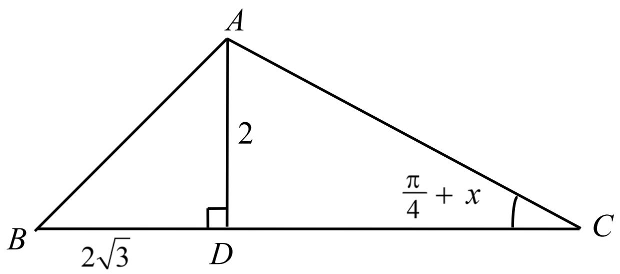
1 A curve has equation $2x+y+2=(x+y)^2+\dfrac{x^2}{1+x^2}$.

Find the equation of the normal to the curve at the point $(0,2)$. [5]

$2x+y+2=(x+y)^2+\dfrac{x^2}{1+x^2}$ $2x+y+2=(x+y)^2+1-\dfrac{1}{1+x^2}\quad ---(1)$ Differentiate (1) w.r.t x: $2+\dfrac{dy}{dx}=2(x+y)\left(1+\dfrac{dy}{dx}\right)-(-1)(1+x^2)^{-2}(2x)\quad ---(2)$ At $(0,2)$, $2+\dfrac{dy}{dx}=2(0+2)\left(1+\dfrac{dy}{dx}\right)+0$ $\dfrac{dy}{dx}=-\dfrac{2}{3}\qquad \Rightarrow$ Gradient of normal is $\dfrac{3}{2}$ Equation of normal at $(0,2)$ is $y=\dfrac{3}{2}x+2$	<u>Be careful when differentiating.</u> Don't forget your chain rule, product rule (or quotient rule), your signs [which can be avoided by adding in appropriate brackets] <u>Be efficient</u> There's no need to make $\dfrac{dy}{dx}$ the subject. You should immediately substitute in $x=0$ and $y=2$

2 The diagram shows triangle ABC, where angle $ACD = \left(\dfrac{\pi}{4}+x\right)$ radians. Point D is on BC such that $AD=2$ and $BD=2\sqrt{3}$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC \approx k(1+\sqrt{3}\ -\ 2x+2x^2),$$

where k is a constant to be determined. [5]

$BC = BD + DC$ $= 2\sqrt{3}+\dfrac{2}{\tan\left(\dfrac{\pi}{4}+x\right)}$ $= 2\sqrt{3}+\dfrac{2}{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$ $= 2\sqrt{3}+\dfrac{2(1-\tan x)}{1+\tan x}$	<u>Secondary school results:</u> Apply TOA in triangle ADC, we have $DC=\dfrac{2}{\tan\left(\dfrac{\pi}{4}+x\right)}$. No need for sine rule or cosine rule. <u>Maclaurin series expansion</u> #1: If x is small, then $2x$, $3x$ are also small. However $(x+a)$ is not

$\approx 2\sqrt{3}+\frac{2(1-x)}{1+x}$ $= 2\sqrt{3}+2(1-x)(1+x)^{-1}$ $= 2\sqrt{3}+2(1-x)[1+(-1)x+\frac{(-1)(-2)}{2!})x^2+...]$ $= 2\sqrt{3}+2(1-x)[1-x+x^2+...]$ $= 2\sqrt{3}+2(1-2x+2x^2+...)$ $\approx 2(1+\sqrt{3}-2x+2x^2)$ where $k=2$ **<u>Alternative</u>** Let $f(x)=BC = BD + DC$ $=2\sqrt{3}+\frac{2}{\tan\left(\frac{\pi}{4}+x\right)}=2\sqrt{3}+2\cot\left(\frac{\pi}{4}+x\right)$ $f'(x)=-2\operatorname{cosec}^2\left(\frac{\pi}{4}+x\right)$ $f''(x)=-2\left[2\operatorname{cosec}\left(\frac{\pi}{4}+x\right)\right]\left[-\operatorname{cosec}\left(\frac{\pi}{4}+x\right)\cot\left(\frac{\pi}{4}+x\right)\right]$ When $x=0$, $f(0)=2\sqrt{3}+2$ $f'(0)=\frac{-2}{\left(\sin\frac{\pi}{4}\right)^2}=-4$ $f''(0)=-2\left(\frac{2}{\sin\frac{\pi}{4}}\right)\left[-\frac{1}{\sin\frac{\pi}{4}}\cot\left(\frac{\pi}{4}\right)\right]=8$ Hence $BC=2\sqrt{3}+2+(-4)x+\frac{8}{2!}x^2+\dots$ $\approx 2\left(\sqrt{3}+1-2x+2x^2\right)$ where $k=2$	considered small, regardless of the size of the constant a. Hence $\tan\left(x+\frac{\pi}{4}\right)\not\approx x+\frac{\pi}{4}$. #2: Since angle x (measured in **radians)** is **small** enough such that x^3 and higher powers of x can be ignored, then $\tan x \approx x$. #3: To find series expansion of $\frac{1}{1+x}$, first rewrite into $(1+x)^{-1}$ then use binomial expansion.

3 **(a)** Find $\int (\ln x)^2 \, dx$. [3]

$\int (\ln x)^2 \, dx = x(\ln x)^2-\int x\cdot 2(\ln x)\cdot\frac{1}{x}\, dx$ $=x(\ln x)^2-2\int \ln x \, dx$ $=x(\ln x)^2-2\left[x\ln x-\int x\cdot\frac{1}{x}\, dx\right]$ $=x(\ln x)^2-2x\ln x+2x+C$	Using by-parts with $u=(\ln x)^2$ and $\frac{dv}{dx}=1$ Hence $\frac{du}{dx}=2(\ln x)\left(\frac{1}{x}\right)$ and $v=x$ 2nd by parts with $u=\ln x$ and $\frac{dv}{dx}=1$ Again don't forget your arbitrary constant.

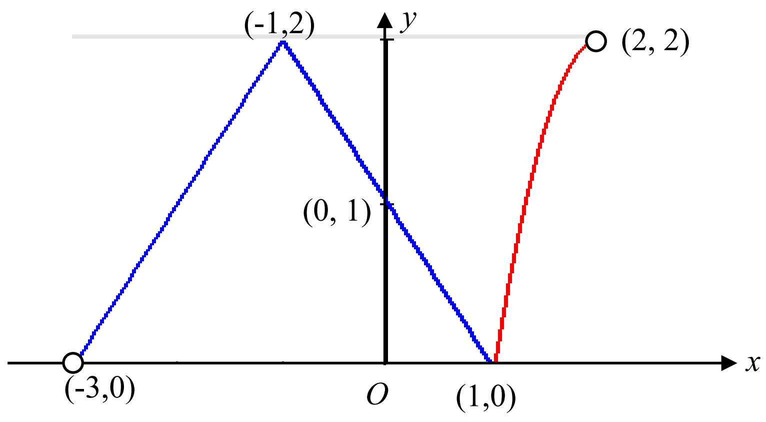
(b) Find $\displaystyle\int \frac{(\sin x+\cos x)^2}{\cos(2x)-2x}\,dx$. [3]

$\displaystyle\int \frac{(\sin x+\cos x)^2}{\cos(2x)-2x}\,dx$ $\displaystyle=\int \frac{\sin^2 x+2\sin x\cos x+\cos^2 x}{\cos(2x)-2x}\,dx$ $\displaystyle=\int \frac{1+\sin 2x}{\cos(2x)-2x}\,dx$ $\displaystyle=-\frac{1}{2}\int \frac{-2\sin 2x-2}{\cos(2x)-2x}\,dx$ $\displaystyle=-\frac{1}{2}\ln\left|\cos(2x)-2x\right|+C$	Sometimes when trigonometric functions are involved, you will need to use trigonometric identity to change the form of the integrand in order to apply integration formula.

4 A curve has equation $y=f(x)$, where

$$f(x)=\begin{cases} 2-|x+1| & \text{for } -3<x\leqslant 1, \\ 2-2(x-2)^2 & \text{for } 1\leqslant x<2. \end{cases}$$

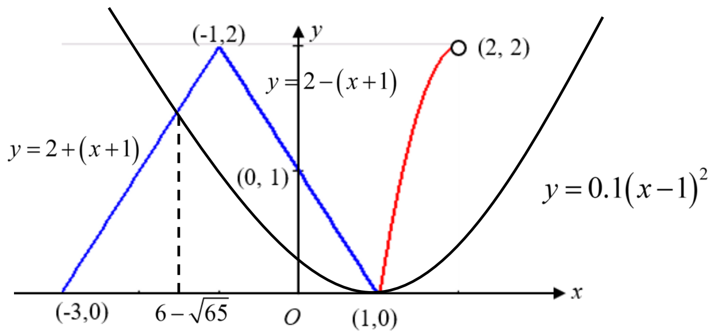
(i) Sketch the curve for $-3<x<2$. [3]



(ii) Hence, solve the inequality $f(x)\leqslant 0.1(x-1)^2$ for $-3<x<2$, leaving your answers in an exact form. [4]

Have to look for points of intersections without using GC

Have to use the graph from (i), to solve the inequality. Hence your solution needs to showcase that method in one way or another.



To find the point of intersection: $2+(x+1)=0.1(x-1)^2$ $10x+30=x^2-2x+1$ $x^2-12x-29=0$ $x=\frac{12\pm\sqrt{260}}{2}$ $x=6+\sqrt{65}$ or $x=6-\sqrt{65}$ (reject) $\therefore f(x)\leqslant 0.1(x-1)^2$ $-3<x\leq 6-\sqrt{65}$ or $x=1$	Recall definition of modulus function: $\lvert x+1\rvert=\begin{cases} x+1, & \text{when } x\geq -1 \\ -(x+1) & \text{when } x<-1\end{cases}$ 2 points of intersection: $x=6-\sqrt{65}$ and $x=1$ Note that $-3<x<2$

5 **Do not use a calculator in answering this question.**
The complex number z satisfies the equation
$$z^2-(4+\mathrm{i})z+2(\mathrm{i}-t)=0,$$
where t is a real number. It is given that one root is of the form $k-k\mathrm{i}$, where k is real and positive. Find t and k, and the other root of the equation. [7]

$(k-k\mathrm{i})^2-(4+\mathrm{i})(k-k\mathrm{i})+2(\mathrm{i}-t)=0$ $\left(k^2-2k(k\mathrm{i})+(k\mathrm{i})^2\right)-\left(4k+k\mathrm{i}-4k\mathrm{i}-k\mathrm{i}^2\right)+2(\mathrm{i}-t)=0$ $\left(k^2-2k^2\mathrm{i}-k^2\right)-\left(4k-3k\mathrm{i}+k\right)+2(\mathrm{i}-t)=0$ $(-5k-2t)+\mathrm{i}\left(2+3k-2k^2\right)=0$ Compare Real part: $-5k-2t=0$ $---(1)$ Imaginary parts: $2+3k-2k^2=0---(2)$ From (2): $-2k^2+3k+2=0$ $(2k+1)(k-2)=0$ $k=-\frac{1}{2}$ or $k=2$ (reject) Substitute $k=2$ into (1), $t=-5$	<u>Secondary school result</u> $(a-b)^2=a^2-2ab+b^2$

$z^2-(4+\mathrm{i})z+2(\mathrm{i}+5)=0$ $z=\dfrac{(4+\mathrm{i})\pm\sqrt{(4+\mathrm{i})^2-4(1)(2\mathrm{i}+10)}}{2}$ $=\dfrac{(4+\mathrm{i})\pm\sqrt{15+8\mathrm{i}-(8\mathrm{i}+40)}}{2}$ $=\dfrac{(4+\mathrm{i})\pm\sqrt{-25}}{2}$ $=\dfrac{4+\mathrm{i}\pm5\mathrm{i}}{2}$ $=2+3\mathrm{i}$ or $2-2\mathrm{i}$ Hence the other root is $2+3\mathrm{i}$ **<u>Alternative (to find other root)</u>** Let $z_1=2-2\mathrm{i}$ and $z_2=$ other root. By sum of roots, $z_1+z_2=-\left(\dfrac{-4-\mathrm{i}}{1}\right)$ $2-2\mathrm{i}+z_2=4+\mathrm{i}$ $z_2=4+\mathrm{i}-2+2\mathrm{i}=2+3\mathrm{i}$	<u>Secondary school result:</u> For any quadratic equation $ax^2+bx+c=0$, Sum of roots, $\alpha+\beta=-\dfrac{b}{a}$ Product of roots, $\alpha\beta=\dfrac{c}{a}$

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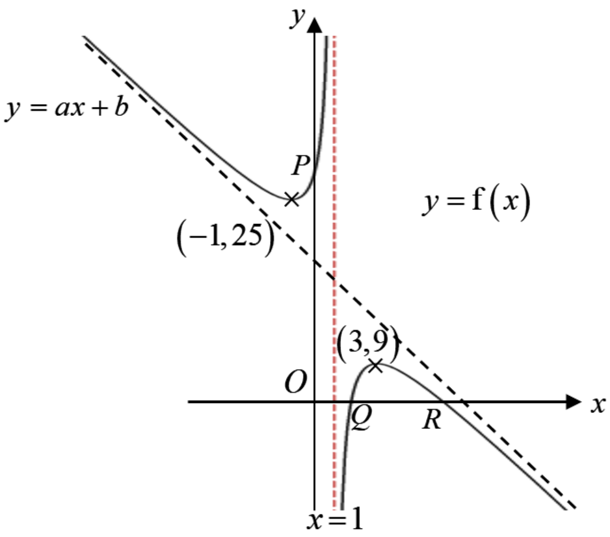


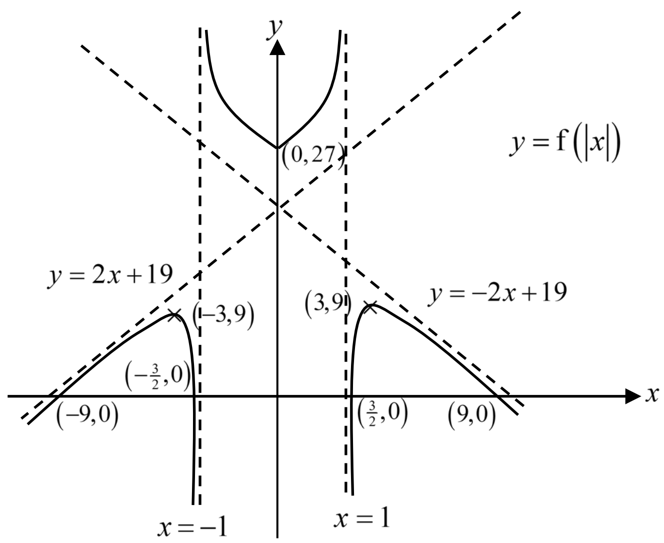
It is given that $\mathrm{f}(x)=ax+b+\dfrac{c}{x-1}$, where a, b and c are constants. The diagram shows the curve with equation $y=\mathrm{f}(x)$. The curve crosses the axes at points P, Q and R, and has stationary points at $(-1,\ 25)$ and $(3,\ 9)$.

Find the values of the constants a, b and c. [4]

$y=\mathrm{f}(x)$ passes through $(3,\ 9)$: $a(3)+b+\dfrac{c}{3-1}=9 \quad \Rightarrow 3a+b+\dfrac{1}{2}c=9---(1)$ $y=\mathrm{f}(x)$ passes through $(-1,25)$: $a(-1)+b+\dfrac{c}{-1-1}=25 \quad \Rightarrow -a+b-\dfrac{1}{2}c=25---(2)$ $\mathrm{f}'(x)=a-\dfrac{c}{(x-1)^2}$ At stationary point $(3,9)$: $a-\dfrac{c}{(3-1)^2}=0 \quad \Rightarrow a-\dfrac{1}{4}c=0---(3)$ Solving (1), (2) and (3), $a=-2, \quad b=19, \quad c=-8$	

It is now given that points P, Q and R have coordinates $(0,27)$, $\left(\dfrac{3}{2},0\right)$ and $(9,0)$ respectively. Sketch the curve

(i) $y=\mathrm{f}(|x|)$, [2]



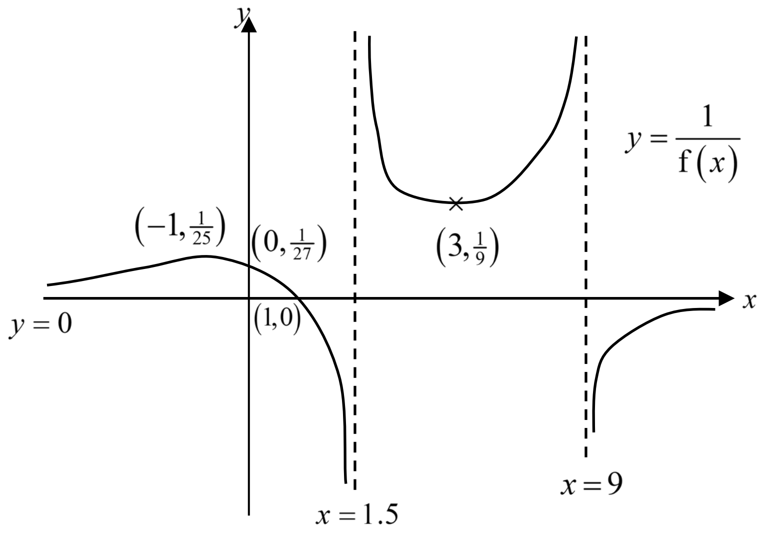
Note that $(0,27)$ is **not** a turning point, hence ensure shape at that point is sketched "sharply"

Observe that the line (oblique asymptote) will be reflected in the y-axis, hence to get the equation of the reflected line, just replace x by $-x$ and we have $y=2x+19$

(ii) $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes, the coordinates of any points where the curve crosses the axes and of any turning point(s).

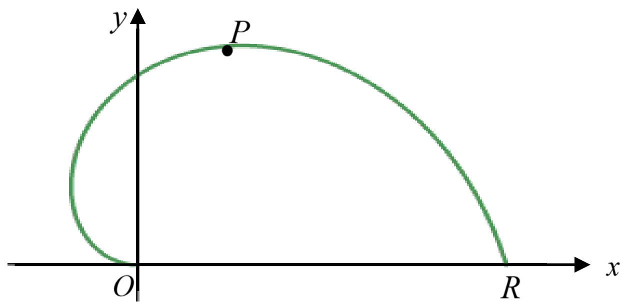
Take note of these requirements which applied to both (i) and (ii).

	$(0,27)\to\left(0,\frac{1}{27}\right)$ $(-1,25)\to\left(-1,\frac{1}{25}\right)$ $(3,9)\to\left(3,\frac{1}{9}\right)$ Take note that $\left(3,\frac{1}{9}\right)$ should be sketched higher than $\left(-1,\frac{1}{25}\right)$

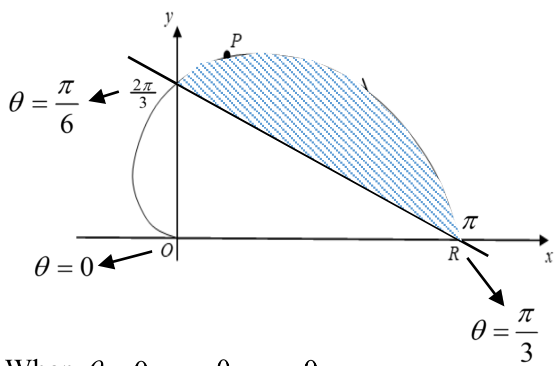
7 The diagram below shows the curve C with parametric equations given by

$$x=-3\theta\cos 3\theta,\quad y=4\theta\sin 3\theta,\qquad \text{for } 0\le\theta\le\frac{\pi}{3}.$$

Point P lies on C with parameter θ and C crosses the x-axis at the origin O and the point R.



(a) Find the area of the region bounded by C and the line $y=\frac{2\pi}{3}-\frac{2x}{3}$, giving your answer correct to 2 decimal places. [4]

 When $\theta=0$, $x=0$, $y=0$ When $\theta=\frac{\pi}{3}$, $x=\pi$, $y=0$ Hence $R(\pi,0)$	First step is to figure out how to add the line $y=\frac{2\pi}{3}-\frac{2x}{3}$ onto the curve in order to visualise the required region, Start by finding the coordinates of the x and y-intercepts of both line and C. Shaded region = region bounded by C and the straight line. Hence area of required region = area bounded by curve from O to R – (area of triangle)

To find y-intercept: Let $x=-3\theta\cos 3\theta=0$ $\cos 3\theta=0$ or $\theta=0$ $3\theta=\frac{\pi}{2}$ $\theta=\frac{\pi}{6}$ When $\theta=\frac{\pi}{6}$, $y=4\left(\frac{\pi}{6}\right)\sin 3\left(\frac{\pi}{6}\right)=\frac{2\pi}{3}$ Hence C cuts the y-axis at O and $\left(0,\frac{2\pi}{3}\right)$. Required area $=\int_0^{\pi} y\,\mathrm{d}x-\frac{1}{2}(\pi)\left(\frac{2\pi}{3}\right)$ $=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(4\theta\sin 3\theta)(9\theta\sin 3\theta-3\cos 3\theta)\,\mathrm{d}\theta-\frac{\pi^2}{3}$ $=2.74$ units^2	$y=4\theta\sin 3\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta}=9\theta\sin 3\theta-3\cos 3\theta$, When $x=0$, ($y=\frac{2\pi}{3}$), hence $\theta=\frac{\pi}{6}$ When $x=\pi$, $\theta=\frac{\pi}{3}$

(b) Use differentiation to find the maximum value of the area of triangle OPR as θ varies, proving that it is a maximum. [6]

Rigor is expected i.e. $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=-50.968$ must be seen. If the first derivative test is used, then $\frac{\mathrm{d}A}{\mathrm{d}\theta}$ has to be evaluated at the 3 values of θ

Note $P(-3\theta\cos 3\theta,\ 4\theta\sin 3\theta)$ Let A be the area of the triangle OPR $A=\frac{1}{2}\times\pi\times 4\theta\sin 3\theta=2\pi\theta\sin 3\theta$ $\frac{\mathrm{d}A}{\mathrm{d}\theta}=2\pi[3\theta\cos 3\theta+\sin 3\theta]$ For max A, $\frac{\mathrm{d}A}{\mathrm{d}\theta}=0$ $2\pi[3\theta\cos 3\theta+\sin 3\theta]=0$ By GC, $\theta=0,0.67625$ or $\theta=0$ (reject) $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=2\pi[-9\theta\sin 3\theta+3\cos 3\theta+3\cos 3\theta]$ $=6\pi[2\cos 3\theta-3\theta\sin 3\theta]$ Substituting $\theta=0.67625$ $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=6\pi[2\cos(3\times 0.67625)-3\theta\sin(3\times 0.67625)]$ $=-50.968<0$ Maximum $A=2\pi\times 0.67625\times\sin(3\times 0.67625)$ $=3.81$ (3s.f.)	Note that O and R are fixed points hence $OR=\pi$

8 **(a)** The sum of the first n terms of a sequence is given by $S_n = \dfrac{n^2+5n}{8}$. Show that the sequence follows an arithmetic progression with common difference d, where d is to be determined.
In a geometric progression, the first term is 100 and its common ratio is $3d$. Find the smallest value of k such that the sum of the first k terms of the arithmetic progression is greater than the sum of the first 30 terms of the geometric progression. [6]

$u_n = S_n - S_{n-1}$ $= \dfrac{n^2+5n}{8} - \dfrac{(n-1)^2+5(n-1)}{8}$ $= \dfrac{1}{8}\left[n^2+5n-\left(n^2-2n+1+5n-5\right)\right]$ $= \dfrac{1}{4}(n+2)$	$u_n = S_n - S_{n-1}$ (not $S_{n+1} - S_n$) is true for all series Be extra careful with algebraic manipulations/expansions, e.g. $-5(n-1) = -5n+5$, not $-5n-5$, not $-5n+1$
Consider $u_n - u_{n-1}$ $= \dfrac{1}{4}(n+2) - \dfrac{1}{4}(n-1+2)$ $= \dfrac{1}{4}$ = constant Hence, the sequence is an arithmetic progression with common difference, $d = \dfrac{1}{4}$	Conclude properly as it is a 'show' question.
$\dfrac{k^2+5k}{8} > \dfrac{100\left(1-\left(\frac{3}{4}\right)^{30}\right)}{1-\frac{3}{4}}$ $k^2+5k-3199.429>0$ $(k-54.119)(k+59.119)>0$ $k<-59.119$ or $k>54.119$ Since $k \geq 0$, smallest $k = 55$	Expression for LHS, i.e. $\dfrac{k^2+5k}{8}$ is already stated in the question. There is no need to rewrite it using $S_n = \dfrac{n}{2}\left[2a+(n-1)d\right]$ Show clearly how the **inequality** is solved Answer is <u>**smallest**</u> $k = 55$, not $k = 55$

(b) The first and second terms of a geometric sequence are $u_1 = a$ and $u_2 = a^2 - a$. If all the terms of the sequence are positive, find the set of values of a for which $\sum_{r=1}^{\infty} u_r$ converges. [2]

Common ratio $r = \frac{u_2}{u_1} = a - 1$ For the series to converge, $\lvert r \rvert < 1$ $\Rightarrow -1 < a - 1 < 1$ $\Rightarrow 0 < a < 2$ Furthermore, for all terms to be positive, $u_1 = a > 0$ and $r = a - 1 > 0$ Alternatively, For the series to converge, <u>and</u> all terms to be positive $0 < r < 1$ $\Rightarrow 0 < a - 1 < 1$ $\Rightarrow 1 < a < 2$ Hence, the set of values of a $\{a \in \mathbb{R} : 1 < a < 2\}$	Read the question carefully, <u>all terms</u> are positive, hence first term and r must be positive. Give your final answer in set notation

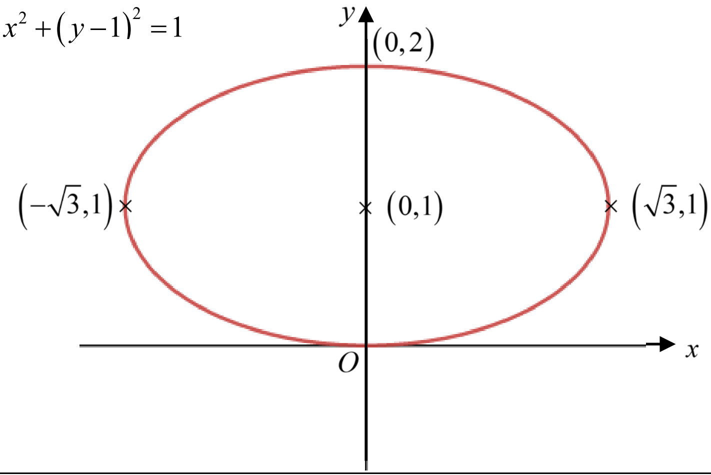
For this sequence, it is known that the sum of all the terms after the nth term is equal to the nth term. Find the value of a and hence the value of $\sum_{r=1}^{\infty} u_r$. [3]

Given : $u_n = u_{n+1} + u_{n+2} + \ldots$ $\Rightarrow ar^{n-1} = ar^n + ar^{n+1} + \ldots.$ $\Rightarrow \frac{ar^n}{r} = \frac{ar^n}{1-r}$ $\Rightarrow r = 1 - r$ $\Rightarrow r = \frac{1}{2}$ i.e. $a = \frac{3}{2}$ Thus $S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$ Alternatively, Using $n = 1$ $S_\infty - a = a$ $\frac{a}{1-(a-1)} - a = a$ $a = \frac{3}{2}$ Thus $S_\infty = \frac{a}{1-r} = \frac{\frac{3}{2}}{1-\frac{1}{2}} = 3$	Read the question carefully, "… sum of all terms <u>**after**</u> the nth term…"

9 The curve C has equation

$$\frac{1}{3}x^2+y^2-2y=0.$$

(i) Sketch C. [2]

$\frac{1}{3}x^2+y^2-2y=0$ $\frac{1}{3}x^2+(y-1)^2-1=0$ $\frac{1}{3}x^2+(y-1)^2=1$ 	Need to label the <u>coordinates</u> of all vertices, including centre. Shape of ellipse is  not  as $\sqrt{3}>1$.

(ii) Use the substitution $x=p\sin\theta$ to show that

$$\int_0^p \sqrt{p^2-x^2}\,\mathrm{d}x = \frac{p^2\pi}{4},$$

where p is a positive constant. [4]

$\int_0^p \sqrt{p^2-x^2}\,\mathrm{d}x$ $=\int_0^{\frac{\pi}{2}} \sqrt{p^2-p^2\sin^2\theta}\,(p\cos\theta)\,\mathrm{d}\theta$ $=p^2\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\,(\cos\theta)\,\mathrm{d}\theta$ $=p^2\int_0^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$ $\left[\cos\theta\geq 0 \text{ for } 0\leq\theta\leq\frac{\pi}{2}\right]$ $=p^2\int_0^{\frac{\pi}{2}}\left(\frac{\cos 2\theta+1}{2}\right)\mathrm{d}\theta$ $=p^2\left[\frac{\sin 2\theta}{4}+\frac{\theta}{2}\right]_0^{\frac{\pi}{2}}$ $=p^2\left(\frac{\sin 2\left(\frac{\pi}{2}\right)}{4}+\frac{\frac{\pi}{2}}{2}\right)=\frac{p^2\pi}{4}$	$x=p\sin\theta$ $\mathrm{d}x=p\cos\theta\,\mathrm{d}\theta$ When $x=0$, $\theta=0$ When $x=p$, $\theta=\frac{\pi}{2}$	Remember to **<u>change the limits</u>** for definite integrals using substitution

(iii) The region R is bounded by C, the line $x=\sqrt{3}$ and the x-axis. Find the exact area of R. [3]

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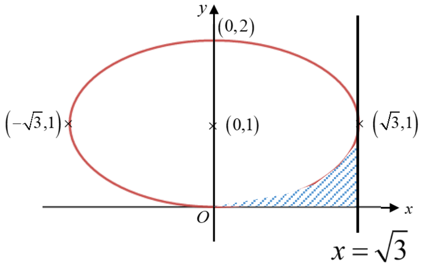

$\frac{1}{3}x^2+(y-1)^2=1$

$y=1-\sqrt{1-\frac{1}{3}x^2} \quad (\because y<1)$

$\text{Area}=\int_0^{\sqrt{3}}\left(1-\sqrt{1-\frac{1}{3}x^2}\right)\mathrm{d}x$

$=\int_0^{\sqrt{3}}1\,\mathrm{d}x-\frac{1}{\sqrt{3}}\int_0^{\sqrt{3}}\sqrt{3-x^2}\,\mathrm{d}x$

$=\sqrt{3}-\frac{1}{\sqrt{3}}\left(\frac{3\pi}{4}\right)$

$=\sqrt{3}-\frac{\sqrt{3}\pi}{4}$
</td><td>
It is very useful to do a simple sketch and shade the region

Do not always assume positive square root for all cases. For this section of the curve, $0\le y\le 1$, so $y=1-\sqrt{1-\frac{1}{3}x^2}$

Always check if you can use the previous part to solve. In this case, $\int_0^{\sqrt{3}}\sqrt{1-\frac{1}{3}x^2}\,\mathrm{d}x$ looks close to the previous result $\int_0^{p}\sqrt{p^2-x^2}\,\mathrm{d}x=\frac{p^2\pi}{4}$

<u>Exact</u> answer is required, so one cannot use GC.
</td></tr>
</table>

(iv) R is rotated completely about the y-axis. Find the exact volume of the solid obtained. [3]

$\text{Volume}=\pi\left(\sqrt{3}\right)^2(1)-\pi\int_0^1 x^2\,\mathrm{d}y$ $=3\pi-\pi\int_0^1\left(6y-3y^2\right)\mathrm{d}y$ $=3\pi-\pi\left[3y^2-y^3\right]_0^1$ $=\pi$	Read the question carefully – R is rotated about **<u>y-axis,</u>** not x-axis. Most students who produced a sketch got this part correct by realising the solid is a **hollow** figure.

(ii) Describe a pair of transformations which transforms the graph of C onto the graph of $x^2+y^2=1$. [2]

Translate 1 units in the negative y-direction. Stretch the resultant curve by a factor of $\frac{1}{\sqrt{3}}$ parallel to the x-axis, y-axis invariant.	

10 Workers are installing zip lines at an adventure park. The points (x, y, z) are defined relative to the entrance at $(0,0,0)$ on ground level, where units are in metres. The ticketing booth at $(100,100,1)$ and lockers at $(200,120,0)$ are also on ground level. Zip lines are laid in straight lines and the widths of zip lines can be neglected. The ground level of the park is modelled as a plane.

(i) Find a cartesian equation of the plane that models the ground level of the park. [2]

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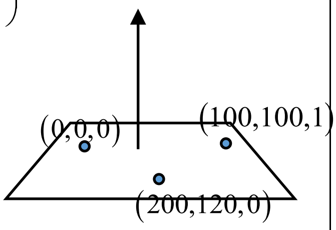
A normal to the ground, $\underset{\sim}{n} = \begin{pmatrix}100\\100\\1\end{pmatrix} \times \begin{pmatrix}200\\120\\0\end{pmatrix}$

$= \begin{pmatrix}-120\\200\\-8000\end{pmatrix}$

$= -40\begin{pmatrix}3\\-5\\200\end{pmatrix}$



Equation of ground: $3x - 5y + 200z = 0$

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Try to avoid unnecessary careless mistakes in cross product by writing clearer and bigger

Question asked for "**<u>cartesian</u>** equation"

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</table>

A zip line connects the points $P(300,120,30)$ and $Q(300,320,25)$, and is modelled as a segment of the line *l*. The façade of a building nearby can be modelled as part of the plane with equation $\mathbf{r}.\begin{pmatrix}1\\-5\\100\end{pmatrix} = 0$. As a safety requirement, every point on the zip line must be at a distance of at least 10 metres away from the façade of the building.

(ii) Write down a vector equation of *l*. Hence, or otherwise, determine if the zip line passes the safety requirement. [4]

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$l: \quad \underset{\sim}{r} = \begin{pmatrix}300\\120\\30\end{pmatrix} + \lambda\begin{pmatrix}0\\-200\\5\end{pmatrix}, \quad \lambda \in \mathbb{R}$

Note that at *P*, $\lambda = 0$; and at *Q*, $\lambda = -1$ and that the origin lies on the plane of the façade.

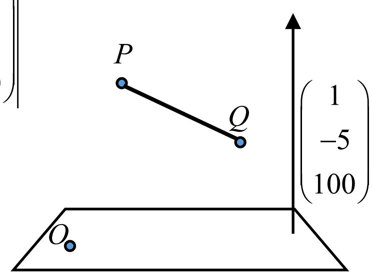
Distance from point on zip line to façade

$= \left|\left[\begin{pmatrix}300\\120\\30\end{pmatrix} + \lambda\begin{pmatrix}0\\-200\\5\end{pmatrix}\right] . \frac{1}{\sqrt{10026}}\begin{pmatrix}1\\-5\\100\end{pmatrix}\right|$

$= \frac{|2700 + 1500\lambda|}{\sqrt{10026}}$

$\geq \frac{1200}{\sqrt{10026}} \qquad \because -1 \leq \lambda \leq 0$

$\approx 11.984 > 10$



Since all points on the zip line are more than 10 m away from the façade of the building, it passes the safety requirement.

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Note the zip line is modelled as a **segment** PQ, not the whole line *l*.

One needs to consider the distance from **a general point on the line segment *PQ*** to the plane.

It is not sufficient to find the distance from *P* (or *Q*) to the plane.

</td>
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</table>

Alternatively,
Distance from P to façade

$$= \left| \begin{pmatrix} 300 \\ 120 \\ 30 \end{pmatrix} . \frac{1}{\sqrt{10026}} \begin{pmatrix} 1 \\ -5 \\ 100 \end{pmatrix} \right|$$

$$= \frac{2700}{\sqrt{10026}}$$

$$= 27.0 > 10$$

Distance from Q to façade

$$= \left| \begin{pmatrix} 300 \\ 320 \\ 25 \end{pmatrix} . \frac{1}{\sqrt{10026}} \begin{pmatrix} 1 \\ -5 \\ 100 \end{pmatrix} \right|$$

$$= \frac{1200}{\sqrt{10026}}$$

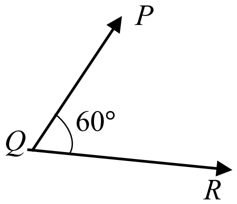
$$= 12.0 > 10$$

Since P and Q are both on the same side of the building façade, all points on the zip line are more than 10 m away from the façade of the building, it passes the safety requirement.

The workers need to install another zip line from Q to $R(127, 220, a)$, where $0 < a < 30$, and the angle PQR is given to be $60°$.

(iii) Find the value of a, leaving your answer to 3 decimal places. [3]

$$\overrightarrow{QP} = \begin{pmatrix} 0 \\ -200 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 0 \\ -40 \\ 1 \end{pmatrix} \text{ and } \overrightarrow{QR} = \begin{pmatrix} -173 \\ -100 \\ a-25 \end{pmatrix}$$



$$\cos 60^\circ = \frac{\begin{pmatrix} 0 \\ -40 \\ 1 \end{pmatrix} . \begin{pmatrix} -173 \\ -100 \\ a-25 \end{pmatrix}}{\sqrt{1601}\sqrt{39929 + (a-25)^2}}$$

$$\left(\frac{1}{4}\right)(1601)\left(39929 + (a-25)^2\right) = \left(4000 + (a-25)\right)^2$$

$$399.25(a-25)^2 - 8000(a-25) - 18417.75 = 0$$

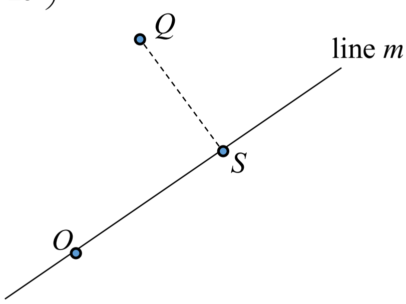
By GC, $a - 25 = -2.08522$ or $a - 25 = 22.1228$ (rej. $\because a < 30$)
$a = 22.915$ (to 3 d.p.)

Recall definition of dot product
$\overrightarrow{QP} \bullet \overrightarrow{QR} = \left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right| \cos\theta$
with <u>both vectors either outward facing or inward facing</u>.

Question reads "to 3 decimal places", so it is a clue to **use GC** to solve an equation with only one unknown.

The façade of the building meets the ground level of the park at line m. A worker sets up a transmitter at point S on line m such that S is nearest to Q.

(iv) Find a vector equation of m and the distance from Q to S. [4]

Façade: $x-5y+100z=0$ Ground: $3x-5y+200z=0$ Solving simultaneously by GC, $m: \ \underset{\sim}{r}=\mu\begin{pmatrix}-50\\10\\1\end{pmatrix}, \quad \mu\in\mathbb{R}$ Since O lies on m, and $\overrightarrow{OQ}=\begin{pmatrix}300\\320\\25\end{pmatrix}$. $QS=\dfrac{\left\lvert\begin{pmatrix}300\\320\\25\end{pmatrix}\times\begin{pmatrix}-50\\10\\1\end{pmatrix}\right\rvert}{\sqrt{2601}}$ $=\dfrac{\left\lvert\begin{pmatrix}70\\-1550\\19000\end{pmatrix}\right\rvert}{\sqrt{2601}}$ $=\dfrac{\sqrt{363407400}}{\sqrt{2601}}$ ≈ 374  The distance from Q to S is 374 m (to 3 s.f.).	Use GC to solve 2 cartesian equations simultaneously Not necessary to add $\begin{pmatrix}0\\0\\0\end{pmatrix}$ in the equation of m. $\underset{\sim}{r}=\cancel{\begin{pmatrix}0\\0\\0\end{pmatrix}}+\mu\begin{pmatrix}-50\\10\\1\end{pmatrix}$

11 A tank contains 500 litres of water in which 100 g of a poisonous chemical called Prokrastenate is dissolved. A solution containing 0.1 g of Prokrastenate per litre is pumped into the tank at a rate of 5 litres per minute, and the well-mixed solution is pumped out at the same rate. By letting x grams be the amount of Prokrastenate in the tank after t minutes, show that

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{k-x}{100},$$

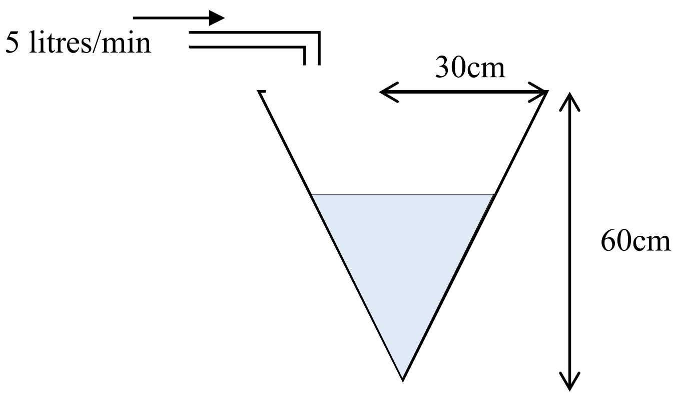
where k is a constant to be determined. [2]

$\dfrac{\mathrm{d}x}{\mathrm{d}t}=0.1\times5-\dfrac{x}{500}\times5$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=0.5-\dfrac{x}{100}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{50-x}{100}$	

Find x in terms of t and find the ~~time taken for a quarter of Prokrastenate to be removed from the tank.~~ value of t when $x = 75$. [5]

$\frac{1}{50-x}\frac{dx}{dt}=\frac{1}{100}$ $\int \frac{1}{50-x}dx=\int \frac{1}{100}dt$ $-\ln\|50-x\|=\frac{t}{100}+C$ $50-x=Ae^{-\frac{t}{100}}$ $x=50-Ae^{-\frac{t}{100}}$ Substituting $t=0, x=100 \Rightarrow A=-50$ Hence, $x=50+50e^{-\frac{t}{100}}$ Substitute $x=75$, $75=50+50e^{-\frac{t}{100}}$ $e^{-\frac{t}{100}}=0.5$ $t=100\ln 2=69.3$	One should always attempt to solve the DE even if k was not found in the earlier part. Evaluate the final answer as 69.3 (3sf) and not leave it as 100ln2.

The well-mixed solution is that is pumped out flows into an empty container, in the form of an open inverted cone with a height of 60 cm and base radius 30 cm, at the same rate (see diagram).



Given that 1 litre = 1000 cm^3,

(i) Show that the volume of the well-mixed solution in the container, V cm^3 can be expressed as $V=\frac{\pi h^3}{12}$, where h cm is the depth of the solution at that instant. [2]

Let the radius and height of water after t min be r cm and h cm respectively. By similar triangles, $\frac{r}{h}=\frac{30}{60}$ $r=\frac{h}{2}$	Need to **explain** $r=\frac{h}{2}$ using similar triangles as it is a 'show' question

$V=\frac{1}{3}\pi r^2 h$ $=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{\pi h^3}{12}$	

(ii) Hence, or otherwise, find the rate of change of the depth of solution after 5 minutes. [4]

[The volume of a cone of base radius r and height h is given by $V=\frac{1}{3}\pi r^2 h$.]

After 5 min, $5\times 5\times 1000=\frac{\pi h^3}{12}$ $h=\sqrt[3]{\frac{30000}{\pi}}=45.708$ $V=\frac{\pi h^3}{12}$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{\pi h^2}{4}$ $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $5000=\frac{\pi(45.708)^2}{4}\times\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=3.0472$ The height of the water level is increasing at the rate 3.05 cm/ min after 5 min.	Convert **5 litres to 5000cm^3** in order to be consistent in the use of units, since cm is used in the question Final answer is in **cm/min**, not cm/s

VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2022

CANDIDATE NAME	

CLASS		INDEX NUMBER	

H2 MATHEMATICS **9758/02**
Paper 2 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)
Writing paper

READ THESE INSTRUCTIONS FIRST

Write your class and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

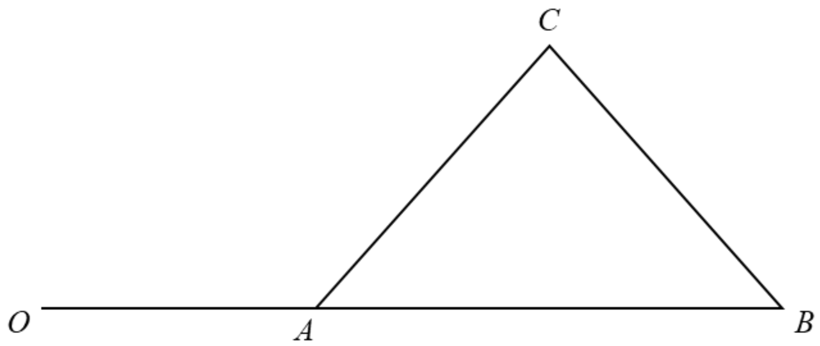
Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of **21** printed pages and **3** blank pages.

Section A: Pure Mathematics [40 marks]

1



Referred to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively.
It is also given that OAB is a straight line (see diagram).

(i) Show that the area of triangle ABC can be written in the form $k\left|(\mathbf{b}-\mathbf{a})\times\mathbf{c}\right|$, where k is a constant to be determined. [3]

It is given that $\overrightarrow{AB}$ is a unit vector and C is equidistant from A and B.

(ii) Give a geometrical interpretation of $\left|(\mathbf{a}-\mathbf{b})\times\mathbf{c}\right|$. [1]

(iii) Show that OB has length $\left|\mathbf{c}.\mathbf{b}-\mathbf{c}.\mathbf{a}\right|+q$, where q is a constant to be determined. [3]

2 **(a)** Functions f and g are defined by

$$\mathrm{f}: x \mapsto 2-\mathrm{e}^{x+a}, \quad \text{for } x \in \mathbb{R},\ x>-2,$$

$$\mathrm{g}: x \mapsto x^2+2x, \quad \text{for } x \in \mathbb{R},\ x<-1,$$

where a is a constant.

(i) Find $\mathrm{g}^{-1}(x)$. [2]

(ii) Explain why the composite function fg exists. [2]

(iii) Find, in terms of a, an expression for $\mathrm{fg}(x)$ and write down the domain of fg. [2]

(iv) Find the range of fg, giving your answer in terms of e and a. [2]

(b) The function h is defined by $\mathrm{h} : x \mapsto \mathrm{e}^{|x+\lambda|}$, $x \in \mathbb{R}$, where λ is a constant.

Does h have an inverse? Justify your answer. [2]

3 **(i)** Show that $\cos\left(x+\frac{\pi}{6}\right)+\cos\left(x-\frac{\pi}{3}\right)=\sqrt{2}\cos\left(x-\frac{\pi}{12}\right)$. [2]

At time t seconds after turning on a switch, the total potential difference across two alternating current power supplies, V, is given by $\mathrm{Re}(z_1+z_2)$, where

$$z_1=2\mathrm{e}^{\left(t+\frac{\pi}{6}\right)\mathrm{i}} \text{ and } z_2=2\mathrm{e}^{\left(t-\frac{\pi}{3}\right)\mathrm{i}}.$$

(ii) Express z_1+z_2 in the form $r\mathrm{e}^{(t-\alpha)\mathrm{i}}$, where $r>0$ and $-\pi<\alpha\leq\pi$, leaving your values of r and α in exact form. [4]

(iii) From the time the switch is turned on, find the amount of time it takes for V to be first at half its maximum value, giving your answer in seconds, correct to 3 decimal places. [2]

(iv) The engineer modified the power supplies so that $z_1 = z_2 = w^{2n}e^{it}$, where $w = 1 + i$ and n is an integer. Show that $V = 2^{n+1}\cos\left(\frac{n\pi}{2} + t\right)$. [3]

4 It is given that $\sum_{r=1}^{N} \ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right] = \ln\left[\frac{(N+1)(N+3)}{2}\right]$.

Use this result to find $\sum_{r=4}^{k+2} \ln\left[\frac{r(r+2)}{3(r-1)(r+1)}\right]$, expressing your answer in the form $\ln\left[\frac{(k+2)(k+4)}{a\left(b^{k-1}\right)}\right]$ where a and b are positive integers to be determined. [5]

A sequence of positive real numbers $v_1, v_2, v_3, \ldots$ is given by $v_1 = 5$ and

$$v_{n+1} = v_n + \sum_{r=1}^{n} \left[(2r+1) + \ln\left[\frac{(r+1)(r+3)}{r(r+2)} \right] \right].$$

Show that $v_{n+1} - v_n = n(n+2) + \ln\left[\frac{(n+1)(n+3)}{2} \right]$. [3]

By considering $\sum_{n=1}^{99} (v_{n+1} - v_n)$, find the numerical value of v_{100}, correct your answer to the nearest integer. [4]

Section B: Probability and Statistics [60 marks]

5 For events A and B, it is given that $\mathrm{P}(A)=0.6$, $\mathrm{P}(B)=0.2$ and $\mathrm{P}(A \mid B')=0.55$. Find

(i) $\mathrm{P}(A \cap B')$, [1]

(ii) $\mathrm{P}(A' \cap B')$. [2]

For a third event C, it is given that $\mathrm{P}(C)=0.4$, $\mathrm{P}(A \cap C)=\mathrm{P}(B \cap C)$, $\mathrm{P}(A' \cap B' \cap C)=0.24$ and $\mathrm{P}(A \cap B \cap C)=0.1$. Determine whether A and C are independent. [3]

6 Four-figure numbers are to be formed from the digits 3, 4, 5, 6, 7 and 8. Find the number of different four-figure numbers that can be formed if

(i) no digit may appear more than once in the number, [1]

(ii) there is at least one repeated digit, but no digit appears more than twice in the number, [3]

(iii) no digit may appear more than once in the number and the sum of all the digits in the number is not divisible by six. [3]

7 An unbiased yellow cubical die has two faces labelled 10, two faces labelled 30 and two faces labelled 50. An unbiased green cubical die has four faces labelled 60, one face labelled 80 and one face labelled 100.

(i) When both dice are thrown, the random variable X is half of the difference between the score on the green die and the score on the yellow die. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4]

(ii) Suppose now the yellow die is replaced by an unbiased blue cubical die with two faces labelled 15, two faces labelled 35 and two faces labelled 55. The random variable W is half of the difference between the score on the green die and the score on the blue die. Without doing further calculation, comment on $\mathrm{E}(W)$ and $\mathrm{Var}(W)$ in relation to $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ respectively. [2]

8 An agricultural experiment was carried out to study the effect of a certain fertilizer on the growth of seedlings. The fertilizer is applied at various concentrations to a random sample of ten plots of land. Seeds are sown and two weeks later, the mean height of the seedlings on each plot of land is measured. The results are shown in the table.

Concentration of fertilizer (x grams/m^2)	5	10	20	30	40	50	60	70	80	90
Mean height of seedlings (y cm)	4.2	9.0	15.6	18.5	19.2	22.5	24.0	25.4	25.4	26.2

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the mean height of seedlings y can be modelled by one of the formulae

$$y = a + bx \quad \text{or} \quad y = c + d \ln x,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y,

(b) $\ln x$ and y. [2]

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + d \ln x$ is the better model. [1]

It is required to estimate the value of x for which $y = 20.0$.

(iv) Explain why neither the regression line of x on y nor the regression line of $\ln x$ on y should be used. [1]

(v) Find the equation of a suitable regression line and use it to find the required estimate. [2]

(vi) Re-write your equation from part **(v)** so that it can be used when y, the mean height of seedlings, is given in mm. [1]

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9 In each batch of manufactured articles, 5% of the articles are found to be defective. A quality inspection is carried out by checking samples of 20 articles.

(i) If 2 or fewer defective articles are found in the sample of 20, the batch is accepted. Find the probability that the batch is accepted. [1]

(ii) Find the least value of n such that the probability of having less than n defective articles in a sample of 20 articles is greater than 0.99. [2]

(iii) Fifty random samples of 20 articles each were taken. Find the probability that the average number of defective articles per sample is at most 1.25. [3]

(iv) It is proposed that a smaller sample be taken for inspection. Find the largest value of k such that the probability of having at least 1 defective article in a sample of k articles is to be less than 0.4? [3]

10 In this question, you should state clearly the values of the parameters of any normal distribution you use.

A meat supplier imports frozen chickens and frozen ducks which are priced by weight. The masses, in kg, of frozen chickens and frozen ducks are modelled as having normal distributions with means and standard deviations as shown in the table.

	Mean mass	Standard deviation
Frozen chickens	1.5	0.3
Frozen ducks	2.6	0.5

(i) Find the probability that a randomly chosen frozen duck has a mass which is more than twice that of a randomly chosen frozen chicken. [3]

The frozen chickens are imported at a cost price of \$6 per kg and the frozen ducks at \$8 per kg.

(ii) The meat supplier orders 100 frozen chickens and 50 frozen ducks in a particular consignment. Find the probability that the meat supplier paid no more than \$2000 for this consignment. State an assumption needed for your calculation. [5]

(iii) A restaurant owner buys frozen ducks from this supplier, who sells the frozen ducks at a profit of 25%. The restaurant owner does not wish to purchase frozen ducks that are too big or too small. If there is a probability of 0.7 that the restaurant owner paid between \$20.00 and \$ a for a randomly chosen frozen duck, find the value of a . [4]

11 **(a)** The average time taken by George to swim 100 m freestyle is 120.05 seconds. He bought a new pair of special goggles from a salesperson who claimed that the goggles will help George swim faster. After wearing the goggles, the time, t seconds, for George to swim 100 m freestyle of each of 50 randomly chosen timings is recorded. The results are summarised as follows.

$$\sum(t-120.05)=-66.4, \quad \sum(t-120.05)^2=1831.945.$$

(i) Test, at the 10% level of significance, whether the goggles helped George to swim faster. You should state your hypotheses and define any symbols you use. [6]

(ii) Explain why this test would be inappropriate if George had taken a random sample of 10 of his 100 m freestyle timings. [1]

(iii) Explain, in the context of the question, the meaning of a "10% significance level". [1]

(b) The random variable X has distribution $\mathrm{N}\left(\mu,\ \sigma^2\right)$.

A random sample of n observations of X is taken, where n is sufficiently large. The mean and variance of this sample is k and 9 respectively.

(i) A test at the 1% level of significance level indicates that the null hypothesis $\mu = 25$ is rejected in favour of the alternative hypothesis $\mu \neq 25$. Find, in terms of n , the range of values of k , giving non exact answers correct to 4 decimal places. [3]

(ii) Hence state the conclusion of the hypothesis test in the case where $k = 24$ and $n = 42$. [1]

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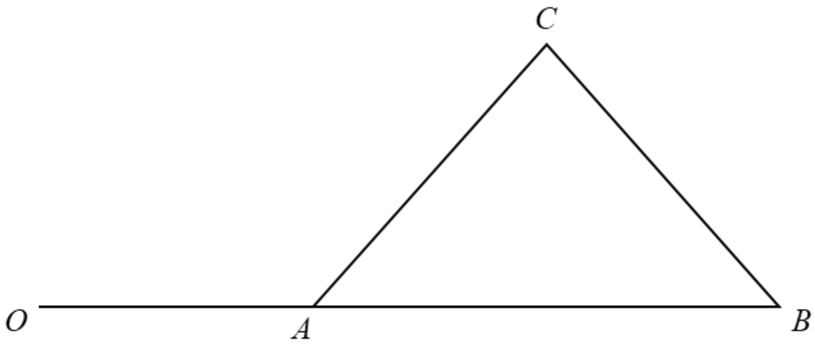
Section A: Pure Mathematics [40 marks]

1



Referred to the origin O, points A, B and C have position vectors **a**, **b** and **c** respectively. It is also given that OAB is a straight line (see diagram).

(i) Show that the area of triangle ABC can be written in the form $k\left|(\mathbf{b}-\mathbf{a})\times\mathbf{c}\right|$, where k is a constant to be determined. [3]

Area $=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$ $=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{c}-\underset{\sim}{a})\right\|$ $=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}-(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{a}\right\|$ $=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}-\underset{\sim}{0}\right\|$ $\quad\because(\underset{\sim}{b}-\underset{\sim}{a})//\underset{\sim}{a}\Rightarrow(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{a}=\underset{\sim}{0}$ $=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}\right\|$ $k=\frac{1}{2}$	Cross product gives vector, not scalar, so $\underset{\sim}{0}$, not 0. Need to write this as this is a 'show' question.
Alternative, Area $=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{CA}\right\|$ $=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times(\underset{\sim}{a}-\underset{\sim}{c})\right\|$ $=\frac{1}{2}\left\|\underset{\sim}{b}\times\underset{\sim}{a}-\underset{\sim}{a}\times\underset{\sim}{a}-\underset{\sim}{b}\times\underset{\sim}{c}+\underset{\sim}{a}\times\underset{\sim}{c}\right\|$ $=\frac{1}{2}\left\|\underset{\sim}{0}-\underset{\sim}{0}-\underset{\sim}{b}\times\underset{\sim}{c}+\underset{\sim}{a}\times\underset{\sim}{c}\right\|$ $\quad\because\underset{\sim}{a}//\underset{\sim}{b}\Rightarrow\underset{\sim}{b}\times\underset{\sim}{a}=\underset{\sim}{0}$ and $\underset{\sim}{a}\times\underset{\sim}{a}=\underset{\sim}{0}$ $=\frac{1}{2}\left\|\underset{\sim}{a}\times\underset{\sim}{c}-\underset{\sim}{b}\times\underset{\sim}{c}\right\|$ $=\frac{1}{2}\left\|(\underset{\sim}{a}-\underset{\sim}{b})\times\underset{\sim}{c}\right\|$ $=\frac{1}{2}\left\|-(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}\right\|=\frac{1}{2}\left\|(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}\right\|$ $k=\frac{1}{2}$	Cross product gives vector, not scalar, so $\underset{\sim}{0}$, not 0. Need to write this as this is a 'show' question. Note that: $(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}\neq\underset{\sim}{c}\times(\underset{\sim}{b}-\underset{\sim}{a})$ Instead, $(\underset{\sim}{b}-\underset{\sim}{a})\times\underset{\sim}{c}=-\underset{\sim}{c}\times(\underset{\sim}{b}-\underset{\sim}{a})$

It is given that $\overrightarrow{AB}$ is a unit vector and C is equidistant from A and B.

(ii) Give a geometrical interpretation of $|(\mathbf{a}-\mathbf{b})\times\mathbf{c}|$. [1]

It represents the perpendicular distance from C to the line (through) AB. OR It represents the height of the triangle ABC, with AB as the base.	$\|(\mathbf{a}-\mathbf{b})\times\mathbf{c}\|$ is a scalar, so think along the line of distances, magnitude, etc… A triangle has 3 heights, state the base of the triangle to distinguish the heights.

(iii) Show that OB has length $|\mathbf{c}.\mathbf{b}-\mathbf{c}.\mathbf{a}|+q$, where q is a constant to be determined. [3]

$OB=\left(\text{length of projection of } \overrightarrow{OC} \text{ on } \overrightarrow{AB}\right)+\frac{AB}{2}$ $=\|\underset{\sim}{c}.(\underset{\sim}{b}-\underset{\sim}{a})\|+\frac{1}{2}$ $=\|\underset{\sim}{c}.\underset{\sim}{b}-\underset{\sim}{c}.\underset{\sim}{a}\|+\frac{1}{2}$	Keep in mind what's given in the question, that $\overrightarrow{AB}$ is a unit vector and C is equidistant from A and B. $(\underset{\sim}{b}-\underset{\sim}{a})=\overrightarrow{AB}$ and hence a unit vector. In dot product, if given a unit vector, think about length of projection.

2 **(a)** Functions f and g are defined by

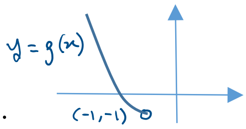
$$\mathrm{f}: x \mapsto 2-\mathrm{e}^{x+a}, \quad \text{for } x\in\mathbb{R},\ x>-2,$$
$$\mathrm{g}: x \mapsto x^2+2x, \quad \text{for } x\in\mathbb{R},\ x<-1,$$

where a is a constant.

(i) Find $\mathrm{g}^{-1}(x)$. [2]

Let $y=\mathrm{g}(x)=x^2+2x$ $x^2+2x-y=0$ $x=\frac{-2\pm\sqrt{4-4(-y)}}{2}=-1\pm\sqrt{1+y}=-1-\sqrt{1+y} \quad (\because x<-1)$ $\therefore \mathrm{g}^{-1}(x)=-1-\sqrt{1+x}$	

(ii) Explain why the composite function fg exists. [2]

Range of g $=(-1,\infty)$. Domain of f $=(-2,\infty)$. Since range of g $\subseteq$ domain of f, fg exists. 	Check if it should be [or (for the interval notation.

(iii) Find, in terms of a, an expression for $\mathrm{fg}(x)$ and write down the domain of fg. [2]

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$\mathrm{fg}(x) = \mathrm{f}(x^2 + 2x)$

$= 2 - \mathrm{e}^{x^2+2x+a}$

$\mathrm{D_{fg}} = \mathrm{D_g} = (-\infty, -1)$.

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(iv) Find the range of fg, giving your answer in terms of e and a. [2]

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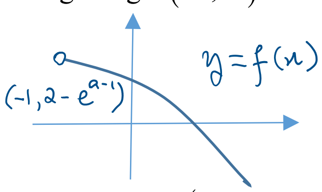
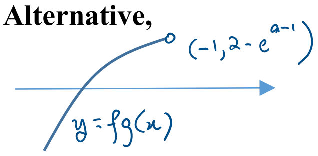
Range of $\mathrm{g} = (-1, \infty)$.



Range of $\mathrm{fg} = (-\infty, 2 - \mathrm{e}^{a-1})$

Alternative,



Range of $\mathrm{fg} = (-\infty, 2 - \mathrm{e}^{a-1})$

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State clearly which graph you are sketching.

Draw the graph for the domain of the function only.

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(b) The function h is defined by $\mathrm{h} : x \mapsto \mathrm{e}^{|x+\lambda|}, \quad x \in \mathbb{R}$, where λ is a constant.

Does h have an inverse? Justify your answer. [2]

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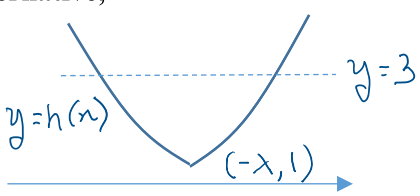
$\mathrm{h}(-\lambda + 1) = \mathrm{e}^{|-\lambda+1+\lambda|} = \mathrm{e}$,

$\mathrm{h}(-\lambda - 1) = \mathrm{e}^{|-\lambda-1+\lambda|} = \mathrm{e}$

Hence, h is not one-one and h does not have an inverse.

Alternative,



The line $y = 3$ cuts the graph of $y = \mathrm{h}(x)$ more than once. Hence, h is not one-one and h does not have an inverse.

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Need to indicate the minimum value of y as it is a critical feature. Draw the line $y = k$ for any $k > 1$.

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3 **(i)** Show that $\cos\left(x+\frac{\pi}{6}\right)+\cos\left(x-\frac{\pi}{3}\right)=\sqrt{2}\cos\left(x-\frac{\pi}{12}\right)$. [2]

$\cos\left(x+\frac{\pi}{6}\right)+\cos\left(x-\frac{\pi}{3}\right)=2\cos\frac{1}{2}\left(x+\frac{\pi}{6}+x-\frac{\pi}{3}\right)\cos\frac{1}{2}\left(x+\frac{\pi}{6}-x+\frac{\pi}{3}\right)$ $=2\cos\left(x-\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)$ $=2\left(\frac{1}{\sqrt{2}}\right)\cos\left(x-\frac{\pi}{12}\right)$ $=\sqrt{2}\cos\left(x-\frac{\pi}{12}\right)$	Refer to MF26 when need to manipulate trigo functions, and look for similar form. For sum of 2 trigo functions, factor formula could be helpful.

At time t seconds after turning on a switch, the total potential difference across two alternating current power supplies, V, is given by $\operatorname{Re}(z_1+z_2)$, where

$$z_1=2\mathrm{e}^{\left(t+\frac{\pi}{6}\right)\mathrm{i}} \text{ and } z_2=2\mathrm{e}^{\left(t-\frac{\pi}{3}\right)\mathrm{i}}.$$

(ii) Express z_1+z_2 in the form $r\mathrm{e}^{(t-\alpha)\mathrm{i}}$, where $r>0$ and $-\pi<\alpha\le\pi$, leaving your values of r and α in exact form. [4]

z_1+z_2 $=2\left(\cos\left(t+\frac{\pi}{6}\right)+\mathrm{i}\sin\left(t+\frac{\pi}{6}\right)\right)+2\left(\cos\left(t-\frac{\pi}{3}\right)+\mathrm{i}\sin\left(t-\frac{\pi}{3}\right)\right)$ $=2\left(\cos\left(t+\frac{\pi}{6}\right)+\cos\left(t-\frac{\pi}{3}\right)\right)+2\mathrm{i}\left(\sin\left(t+\frac{\pi}{6}\right)+\sin\left(t-\frac{\pi}{3}\right)\right)$ $=2\sqrt{2}\cos\left(t-\frac{\pi}{12}\right)+2\mathrm{i}\left(2\sin\left(t-\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)\right)$ $=2\sqrt{2}\cos\left(t-\frac{\pi}{12}\right)+\mathrm{i}\frac{4}{\sqrt{2}}\sin\left(t-\frac{\pi}{12}\right)$ $=2\sqrt{2}\left[\cos\left(t-\frac{\pi}{12}\right)+\mathrm{i}\sin\left(t-\frac{\pi}{12}\right)\right]$ $=2\sqrt{2}\mathrm{e}^{\left(t-\frac{\pi}{12}\right)\mathrm{i}}$	For addition of 2 complex numbers, we will consider either cartesian or trigo form. Since answer is required in exp form, we work with trigo form. Also, bear in mind what you have done in **part (i),** which involves trigo and the angles are the same.

(iii) From the time the switch is turned on, find the amount of time it takes for V to be first at half its maximum value, giving your answer in seconds, correct to 3 decimal places. [2]

$V=\text{Re}(z_1+z_2)=2\sqrt{2}\cos\left(t-\frac{\pi}{12}\right)$ Maximum value of $V=2\sqrt{2}$ When V is half its maximum value, $\sqrt{2}=2\sqrt{2}\cos\left(t-\frac{\pi}{12}\right)$ $\cos\left(t-\frac{\pi}{12}\right)=\frac{1}{2}$ Hence, smallest positive value of t is such that $t-\frac{\pi}{12}=\frac{\pi}{3}\Rightarrow t=\frac{5\pi}{12}=1.309$ (to 3 d.p.). The time taken is 1.309 s.	Recall $-1\le\cos\theta\le 1$ $-A\le A\cos\theta\le A$ Max value of $A\cos\theta=A$ Min value of $A\cos\theta=-A$

(iv) The engineer modified the power supplies so that $z_1=z_2=w^{2n}e^{it}$, where $w=1+i$ and n is an integer. Show that $V=2^{n+1}\cos\left(\frac{n\pi}{2}+t\right)$. [3]

$z_1=z_2=w^{2n}e^{it}\Rightarrow z_1+z_2=2w^{2n}e^{it}$ $\lvert z_1+z_2\rvert=\lvert 2w^{2n}e^{it}\rvert$ $=2\lvert 1+i\rvert^{2n}\lvert e^{it}\rvert$ $=2\left(\sqrt{2}\right)^{2n}$ $=2^{n+1}$ $\arg(z_1+z_2)=\arg\left(2w^{2n}e^{it}\right)$ $\arg(2)+2n\arg(w)+\arg\left(e^{it}\right)=0+2n\arg(1+i)+\arg\left(e^{it}\right)$ $=2n\left(\frac{\pi}{4}\right)+t$ $=\frac{n\pi}{2}+t$ Hence, $V=\text{Re}(z_1+z_2)=2^{n+1}\cos\left(\frac{n\pi}{2}+t\right)$. Alternatively, $z_1+z_2=2w^{2n}e^{it}$ $=2(1+i)^{2n}e^{it}$ $=2\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^{2n}e^{it}$ $=2\left(\sqrt{2}\right)^{2n}e^{\frac{in\pi}{2}}e^{it}$ $=2^{n+1}e^{i\left(\frac{n\pi}{2}+t\right)}$ $=2^{n+1}\left(\cos\left(\frac{n\pi}{2}+t\right)+i\sin\left(\frac{n\pi}{2}+t\right)\right)$ $V=\text{Re}(z_1+z_2)=2^{n+1}\cos\left(\frac{n\pi}{2}+t\right)$	$2w^{2n}e^{it}$ is multiplication of 2 complex numbers. Consider changing everything to exp form

4 It is given that $\sum_{r=1}^{N} \ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right] = \ln\left[\frac{(N+1)(N+3)}{2}\right]$.

Use this result to find $\sum_{r=4}^{k+2} \ln\left[\frac{r(r+2)}{3(r-1)(r+1)}\right]$, expressing your answer in the form $\ln\left[\frac{(k+2)(k+4)}{a\left(b^{k-1}\right)}\right]$ where a and b are positive integers to be determined. [5]

$\sum_{r=4}^{k+2} \ln\left[\frac{r(r+2)}{3(r-1)(r+1)}\right]$ $=\sum_{r=4}^{k+2}\left[\ln\left[\frac{r(r+2)}{(r-1)(r+1)}\right]-\ln 3\right]$ $=\sum_{r=3}^{k+1} \ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]-\sum_{r=4}^{k+2}\ln 3$ $=\sum_{r=1}^{k+1} \ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]-\sum_{r=1}^{2} \ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]-(k-1)\ln 3$ $=\ln\left[\frac{(k+2)(k+4)}{2}\right]-\ln\left[\frac{(3)(5)}{2}\right]-(k-1)\ln 3$ $=\ln\left[\frac{(k+2)(k+4)}{15\left(3^{k-1}\right)}\right]$	Note: $\ln\left(\frac{a}{b}\right)=\ln a-\ln b$ So $\ln\left(\frac{a}{3}\right)=\ln a-\ln 3$ Do revise the logarithm properties as they are assumed O Level knowledge.

A sequence of positive real numbers $v_1, v_2, v_3, \ldots$ is given by

$$v_1 = 5 \text{ and } v_{n+1} = v_n + \sum_{r=1}^{n}\left[(2r+1)+\ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]\right].$$

Show that $v_{n+1}-v_n = n(n+2)+\ln\left[\frac{(n+1)(n+3)}{2}\right]$. [3]

$v_{n+1}-v_n=\sum_{r=1}^{n}\left[(2r+1)+\ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]\right]$ $=\sum_{r=1}^{n}(2r+1)+\sum_{r=1}^{n}\ln\left[\frac{(r+1)(r+3)}{r(r+2)}\right]$ $=\frac{n}{2}(3+2n+1)+\ln\left[\frac{(n+1)(n+3)}{2}\right]$ $=n(n+2)+\ln\left[\frac{(n+1)(n+3)}{2}\right]$	A 'show' question, working needs to be rigorous. Write down the formula for sum of AP with substitution of values.

By considering $\sum_{n=1}^{99}\left(v_{n+1}-v_n\right)$, find the numerical value of v_{100}, correct your answer to the nearest integer. [4]

$\sum_{n=1}^{99}\left(v_{n+1}-v_n\right)=\sum_{n=1}^{99}\left(n(n+2)+\ln\left[\frac{(n+1)(n+3)}{2}\right]\right)$ $=338916.3055$ by GC $\sum_{n=1}^{99}\left(v_{n+1}-v_n\right)=v_2-v_1$ $+v_3-v_2$ $+v_4-v_3$ $+\ldots$ $+v_{99}-v_{98}$ $+v_{100}-v_{99}$ $=v_{100}-v_1$ $=v_{100}-5$ Hence, $v_{100}-5=338916.3055$ $v_{100}=338916.3055+5=338921$ (to nearest integer)	Read question carefully for all the information provided. Need to find v_{100} using $\sum_{n=1}^{99}\left(v_{n+1}-v_n\right)$ Given $v_1=5$ and showed $v_{n+1}-v_n=n(n+2)+\ln\left[\frac{(n+1)(n+3)}{2}\right]$ in previous part. Think of how to link all these together

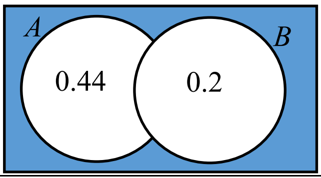
Section B: Probability and Statistics [60 marks]

5 For events A and B, it is given that $\mathrm{P}(A)=0.6$, $\mathrm{P}(B)=0.2$ and $\mathrm{P}(A\mid B')=0.55$. Find

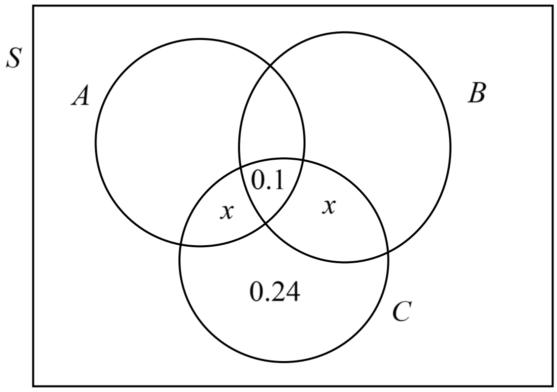
(i) $\mathrm{P}(A\cap B')$, [1]

$\mathrm{P}(A\cap B')=\mathrm{P}(A\mid B')\times\mathrm{P}(B')$ $=0.55\times(1-0.2)$ $=0.44$	

(ii) $\mathrm{P}(A'\cap B')$. 2]

$\mathrm{P}(A'\cap B')=1-\mathrm{P}(A\cup B)$ $=1-\left(\mathrm{P}(A\cap B')+\mathrm{P}(B)\right)$ $=1-(0.44+0.2)$ $=0.36$	 	Use a Venn diagram. Fill in the prob given and found in earlier part.

For a third event C, it is given that $\mathrm{P}(C)=0.4$, $\mathrm{P}(A\cap C)=\mathrm{P}(B\cap C)$, $\mathrm{P}(A'\cap B'\cap C)=0.24$ and $\mathrm{P}(A\cap B\cap C)=0.1$. Determine whether A and C are independent. [3]

 Since $\mathrm{P}(A\cap C)=\mathrm{P}(B\cap C)$, Let $\mathrm{P}(A\cap C\cap B')=\mathrm{P}(B\cap C\cap A')=x$ $\mathrm{P}(C)=0.1+0.24+2x=0.4 \quad \Rightarrow x=0.03$ $\mathrm{P}(A\cap C)=0.1+0.03=0.13\neq \mathrm{P}(A)\times\mathrm{P}(C)=0.6\times0.4=0.24$ $\therefore A$ and C are not independent.	Use a Venn diagram. Fill in the prob given and found in earlier part.

6 Four-figure numbers are to be formed from the digits 3, 4, 5, 6, 7 and 8. Find the number of different four-figure numbers that can be formed if

(i) no digit may appear more than once in the number, [1]

No. of numbers = ${}^6C_4\times4!={}^6P_4=360$	B1: 360

(ii) there is at least one repeated digit, but no digit appears more than twice in the number, [3]

Case 1: **AABC** no. of numbers = ${}^6C_1\times{}^5C_2\times\dfrac{4!}{2!}=720$ Case 2: **AABB** no. of numbers = ${}^6C_2\times\dfrac{4!}{2!2!}=90$ Total no. of numbers = 720+90 =810 Alternatively, Total – all different – all same – 3 same and 1 different $=1296-360-6-{}^6C_1\times{}^5C_1\times\dfrac{4!}{3!}=810$	'at least', 'more than' suggest that there a few cases to consider. List out the cases systematically. 1 repeated digit, 2 repeated digit … Be clear minded when considering complementary method. Always ask if you have considered all the cases to exclude.

(iii) no digit may appear more than once in the number and the sum of all the digits in the number is not divisible by six. [3]

No. of numbers = $360-3\times4!=288$	Use complement with answer from (i) as there are fewer cases for number divisible by 6. To find number that are divisible by 6, be systematic so that you do not miss out cases. To get 4 digits from 3, 4, 5, 6, 7 and 8, Smallest possible sum = 3+4+5+6 = 18 (1 case only) Largest possible sum = 5+6+7+8 = 26 So the sum that is divisible by 6 can only be 18 or 24. To get 24, largest possible sum need to subtract 2, i.e 3,6,7,8 or 5,4,7,8 $\therefore$ 3 cases: 3,4,5,6 or 3,6,7,8 or 5,4,7,8. Each has 4! ways to arrange the digits.

7 An unbiased yellow cubical die has two faces labelled 10, two faces labelled 30 and two faces labelled 50. An unbiased green cubical die has four faces labelled 60, one face labelled 80 and one face labelled 100.

(i) When both dice are thrown, the random variable X is half of the difference between the score on the green die and the score on the yellow die. Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4]

Green \ Yellow	10	30	50
60	$\frac{50}{2}$	$\frac{30}{2}$	$\frac{10}{2}$
80	$\frac{70}{2}$	$\frac{50}{2}$	$\frac{30}{2}$
100	$\frac{90}{2}$	$\frac{70}{2}$	$\frac{50}{2}$

x	5	15	25	35	45
$\mathrm{P}(X=x)$	$\frac{4}{6}\times\frac{2}{6}=\frac{2}{9}$	$\frac{5}{18}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{18}$

Always check that the probabilities sum to 1.

$$\mathrm{E}(X)=\frac{2}{9}(5)+\frac{5}{18}(15)+\frac{1}{3}(25)+\frac{1}{9}(35)+\frac{1}{18}(45)$$
$$=20$$
$$\mathrm{E}(X^2)=\frac{2}{9}(5)^2+\frac{5}{18}(15)^2+\frac{1}{3}(25)^2+\frac{1}{9}(35)^2+\frac{1}{18}(45)^2$$
$$=525$$
$$\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2=525-20^2=125$$

Alternative Solution Let Y and G be the score on the yellow die and green die respectively.	

y	10	30	50
$P(Y=y)$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{2}{6}$

g	60	80	100
$P(G=g)$	$\frac{4}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(G)=\frac{2}{3}(60)+\frac{1}{6}(80)+\frac{1}{6}(100)=70$$
$$E(G^2)=\frac{2}{3}(60)^2+\frac{1}{6}(80)^2+\frac{1}{6}(100)^2=\frac{15400}{3}$$
$$\text{Var}(G)=E(G^2)-[E(G)]^2=\frac{15400}{3}-70^2=\frac{700}{3}$$

$$E(Y)=30 \quad (\text{by symmetry})$$
$$E(Y^2)=\frac{2}{6}(10)^2+\frac{2}{6}(30)^2+\frac{2}{6}(50)^2=\frac{3500}{3}$$
$$\text{Var}(Y)=E(Y^2)-[E(Y)]^2=\frac{3500}{6}-30^2=\frac{800}{3}$$

$$X=\frac{1}{2}|G-Y|=\frac{1}{2}(G-Y) \qquad \because G-Y \text{ is always positive}$$
$$E(X)=E\left(\frac{G-Y}{2}\right)=\frac{1}{2}[E(G)-E(Y)]=\frac{1}{2}[70-30]=20$$
$$\text{Var}(X)=\left(\frac{1}{2}\right)^2[\text{Var}(G)+\text{Var}(Y)]$$
$$=\left(\frac{1}{2}\right)^2\left[\frac{700}{3}+\frac{800}{3}\right]=125$$

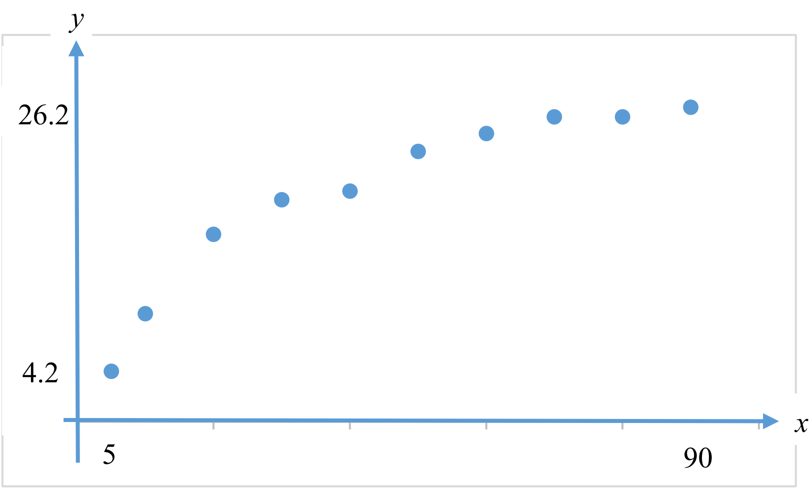
(ii) Suppose now the yellow die is replaced by an unbiased blue cubical die with two faces labelled 15, two faces labelled 35 and two faces labelled 55. The random variable W is half of the difference between the score on the green die and the score on the blue die. Without doing further calculation, comment on $E(W)$ and $\text{Var}(W)$ in relation to $E(X)$ and $\text{Var}(X)$ respectively. [2]

Observe that the number on blue die is obtain by adding 5 to every number on the yellow die. Hence, $W=X-2.5$. Therefore $E(W)$ is 2.5 less than $E(X)$. Adding 5 to all the numbers on the yellow die does not affect the spread of the data, hence $\text{Var}(W)$ is the same as $\text{Var}(X)$	

8 An agricultural experiment was carried out to study the effect of a certain fertilizer on the growth of seedlings. The fertilizer is applied at various concentrations to a random sample of ten plots of land. Seeds are sown and two weeks later, the mean height of the seedlings on each plot of land is measured. The results are shown in the table.

Concentration of fertilizer (x grams/m^2)	5	10	20	30	40	50	60	70	80	90
Mean height of seedlings (y cm)	4.2	9.0	15.6	18.5	19.2	22.5	24.0	25.4	25.4	26.2

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

	Label minimum and maximum x and y values. Check that the 8th and 9th points are level as they have the same y-value.

It is thought that the mean height of seedlings y can be modelled by one of the formulae

$$y = a + bx \quad \text{or} \quad y = c + d\ln x,$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y,

(b) $\ln x$ and y. [2]

(a) $r = 0.925588 \approx 0.9256$ **(b)** $r = 0.996583 \approx 0.9966$	

(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = a + bx$ or $y = c + d\ln x$ is the better model. [1]

From the scatter diagram, it is observed that as x increases, y increases by decreasing amounts, and the product moment correlation coefficient between $\ln x$ and y is closer to 1 than that of x and y. Hence $y = c + d\ln x$ is the better model.	Use both scatter diagram and correlation coefficient to compare.

It is required to estimate the value of x for which $y = 20.0$.

(iv) Explain why neither the regression line of x on y nor the regression line of $\ln x$ on y should be used. [1]

Since the fertilizer is applied at various concentrations, x (and hence $\ln x$) is the predetermined/controlled/independent variable. Thus, neither the regression line of x on y nor the regression line of $\ln x$ on y should be used. To estimate x given $y = 20.0$, the regression line of y on $\ln x$ should be used.	

(v) Find the equation of a suitable regression line and use it to find the required estimate. [2]

From GC, the equation of the regression line of y on $\ln x$ is $y = -8.4492 + 7.8126\ln x$ $\therefore y = -8.45 + 7.81\ln x$ When $y = 20.0$, $x = 38.147 \approx 38$	

(vi) Re-write your equation from part **(v)** so that it can be used when y, the mean height of seedlings, is given in mm. [1]

Replace y by $0.1y$, $0.1y = -8.4492 + 7.8126\ln x$ $\therefore y = -84.5 + 78.1\ln x$	1 cm = 10 mm

9 In each batch of manufactured articles, 5% of the articles are found to be defective. A quality inspection is carried out by checking samples of 20 articles.

(i) If 2 or fewer defective articles are found in the sample of 20, the batch is accepted. Find the probability that the batch is accepted. [1]

Let Y be the number of defective articles out of 20. $Y \sim \mathrm{B}(20, 0.05)$ Required probability $= \mathrm{P}(Y \le 2) \approx 0.925$	

(ii) Find the least value of n such that the probability of having less than n defective articles in a sample of 20 articles is greater than 0.99. [2]

$\mathrm{P}(Y < n) > 0.99$ $\mathrm{P}(Y \le n-1) > 0.99$ From GC, $\mathrm{P}(Y \le 3) = 0.98410 < 0.99$ $\mathrm{P}(Y \le 4) = 0.99743 > 0.99$ $\therefore$ least value of $n = 5$	For your working, check that $\mathrm{P}(Y \le 3) < 0.99$ and $\mathrm{P}(Y \le 4) > 0.99$ to check that least value of $n-1$ is 4 (or draw a table to check).

(iii) Fifty random samples of 20 articles each were taken. Find the probability that the average number of defective articles per sample is at most 1.25. [3]

Let $\bar{Y} = \dfrac{Y_1 + Y_2 + \ldots + Y_{50}}{50}$ $E(\bar{Y}) = E(Y) = 20 \times 0.05 = 1$ $\text{Var}(\bar{Y}) = \dfrac{\text{Var}(Y)}{50} = \dfrac{20 \times 0.05 \times 0.95}{50} = \dfrac{0.95}{50}$ By CLT, $\bar{Y} \sim N\left(1, \dfrac{0.95}{50}\right)$ approx $P(\bar{Y} \le 1.25) = 0.96514 \approx 0.965$	

(iv) It is proposed that a smaller sample be taken for inspection. Find the largest value of k such that the probability of having at least 1 defective article in a sample of k articles is to be less than 0.4? [3]

Let W be the number of defective articles out of k. $W \sim B(k, 0.05)$ $P(W \ge 1) < 0.4$ $1 - P(W = 0) < 0.4$ $P(W = 0) > 0.6$ $(0.95)^k > 0.6$ $k < \dfrac{\ln 0.6}{\ln 0.95} = 9.9589$ $\therefore$ largest value of $k = 9$	

10 In this question, you should state clearly the values of the parameters of any normal distribution you use.

A meat supplier imports frozen chickens and frozen ducks which are priced by weight. The masses, in kg, of frozen chickens and frozen ducks are modelled as having normal distributions with means and standard deviations as shown in the table.

	Mean mass	Standard deviation
Frozen chickens	1.5	0.3
Frozen ducks	2.6	0.5

(i) Find the probability that a randomly chosen frozen duck has a mass which is more than twice that of a randomly chosen frozen chicken. [3]

Let X kg and Y kg be the mass of a frozen chicken and a frozen duck respectively. $X \sim N(1.5, 0.3^2)$ and $Y \sim N(2.6, 0.5^2)$ $Y - 2X \sim N(-0.4, 0.61)$ $P(Y > 2X) = P(Y - 2X > 0)$ $= 0.30427$ ≈ 0.304	

The frozen chickens are imported at a cost price of \$6 per kg and the frozen ducks at \$8 per kg.

(ii) The meat supplier orders 100 frozen chickens and 50 frozen ducks in a particular consignment. Find the probability that the meat supplier paid no more than \$2000 for this consignment. State an assumption needed for your calculation. [5]

Let $T = 6(X_1 + X_2 + ... + X_{100}) + 8(Y_1 + Y_2 + ... + Y_{50})$ $E(T) = 6(100)(1.5) + 8(50)(2.6) = 1940$ $Var(T) = 6^2(100)(0.3^2) + 8^2(50)(0.5^2) = 1124$ $\therefore T \sim N(1940, 1124)$ $P(T \le 2000) = 0.96324$ ≈ 0.963 The masses of **<u>all</u>** frozen chickens and frozen ducks are independent of one another.	Assumption: For the calculations of $E(T)$ and $Var(T)$ to be valid, the random variables $X_1,...,X_{100},Y_1,...,Y_{50}$ must be independent of one another. Then write this in context.

(iii) A restaurant owner buys frozen ducks from this supplier, who sells the frozen ducks at a profit of 25%. The restaurant owner does not wish to purchase frozen ducks that are too big or too small. If there is a probability of 0.7 that the restaurant owner paid between \$20.00 and \$$a$ for a randomly chosen frozen duck, find the value of a. [4]

Let \$$W$ be the selling price of a randomly chosen frozen duck. $W = 1.25(8)Y = 10Y$ $W \sim N(10 \times 2.6, 10^2 \times 0.5^2)$ i.e. $W \sim N(26, 25)$ $P(20 < W < a) = 0.7$ $P(W < a) - P(W < 20) = 0.7$ $P(W < a) = 0.7 + P(W < 20)$ $= 0.81507$ $\therefore a = 30.48$ **<u>Alternatively,</u>** Let \$$W$ be the selling price of a randomly chosen frozen duck. $W = 1.25(8)Y = 10Y$ $P(20 < 10Y < a) = 0.7$ $P\left(2 < Y < \frac{a}{10}\right) = 0.7$ $P\left(Y < \frac{a}{10}\right) - P(Y < 2) = 0.7$ $P\left(Y < \frac{a}{10}\right) = 0.7 + P(Y < 2) = 0.81507$	Leave final answer for money to 2 decimal places.

$\therefore \frac{a}{10} = 3.04837$ $\Rightarrow a = 30.48$	

11 (a) The average time taken by George to swim 100 m freestyle is 120.05 seconds. He bought a new pair of special goggles from a salesperson who claimed that the goggles will help George swim faster. After wearing the goggles, the time, t seconds, for George to swim 100 m freestyle of each of 50 randomly chosen timings is recorded. The results are summarised as follows.

$$\sum(t-120.05) = -66.4, \quad \sum(t-120.05)^2 = 1831.945.$$

(i) Test, at the 10% level of significance, whether the goggles helped George to swim faster. You should state your hypotheses and define any symbols you use. [6]

Let T seconds be the time taken for George to swim the 100m freestyle and let μ be the population mean of T. $H_0: \mu = 120.05$ $H_1: \mu < 120.05$ Level of significance: 10% Test Statistic: Since n = 50 is sufficiently large, by Central Limit Theorem, $\overline{T}$ is approximately normally distributed. When H_0 is true, $Z = \frac{\overline{T}-120.05}{S/\sqrt{n}} \sim N(0,1)$ approximately Computation: $n = 50$, $\bar{t} = \frac{-66.4}{50} + 120.05 = 118.722$ $s^2 = \frac{1}{49}\left(1831.945 - \frac{(-66.4)^2}{50}\right) \approx 35.5871$ p – value $\approx 0.05773 = 0.0577$ (3s.f) Conclusion: Since p – value $= 0.0577 < 0.1$, H_0 is rejected at the 10% level of significance. So, there is sufficient evidence to conclude that the mean time taken to swim 100 m freestyle is less than 120.05 s.	If the variable used is t, define T as the random variable, and define μ as the population mean of T. Test statistic should have the value of $\mu = 120.05$ substituted – it is when H_0 is true. For final sentence: Remember to conclude that there is sufficient/insufficient evidence to conclude 'H_1' (**in context of the question**).

(ii) Explain why this test would be inappropriate if George had taken a random sample of 10 of his 100 m freestyle timings. [1]

The time taken for George to swim the 100 m freestyle is not known to be normally distributed. If a sample of 10 of his 100 m freestyle timings is taken, <u>Central Limit Theorem cannot be applied</u> to approximate <u>sample mean time taken,</u> $\overline{T}$, for George to swim the 100 m freestyle to a normal distribution. Hence the test would not be appropriate.	

(iii) Explain, in the context of the question, the meaning of a "10% significance level". [1]

A 10% significance level means that there is a probability of 0.1 that the test concludes that the mean time taken for George to swim 100 m freestyle is less than 120.05 seconds, when it is actually 120.05 seconds.	… that the test concludes 'H_1', when 'H_0 is actually true' (**in context**).

(b) The random variable X has distribution $N(\mu,\ \sigma^2)$.

A random sample of n observations of X is taken, where n is sufficiently large. The mean and variance of this sample is k and 9 respectively.

(i) A test at the 1% level of significance level indicates that the null hypothesis $\mu = 25$ is rejected in favour of the alternative hypothesis $\mu \neq 25$. Find, in terms of n, the range of values of k, giving non exact answers correct to 4 decimal places. [3]

$H_0: \mu = 25$ $H_1: \mu \neq 25$ Level of significance: 1% Test Statistic: when H_0 is true $Z = \dfrac{\overline{X} - 25}{S/\sqrt{n}} \sim N(0,1)$ approximately Rejection region: $z \leq -2.57583$ or $z \geq 2.57583$ Computation: $\overline{x} = k$, $s^2 = \dfrac{n}{n-1} \times 9$ $z - \text{calculated} = \dfrac{k-25}{\frac{s}{\sqrt{n}}} = \dfrac{k-25}{\frac{\frac{3\sqrt{n}}{\sqrt{n-1}}}{\sqrt{n}}} = \dfrac{k-25}{\frac{3}{\sqrt{n-1}}}$ Conclusion: H_0 is rejected at 1% significance level $\Rightarrow \dfrac{k-25}{\frac{3}{\sqrt{n-1}}} \leq -2.57583$ or $\dfrac{k-25}{\frac{3}{\sqrt{n-1}}} \geq 2.57583$ $\Rightarrow k \leq 25 - \dfrac{7.7275}{\sqrt{n-1}}$ or $k \geq 25 + \dfrac{7.7275}{\sqrt{n-1}}$	

(ii) Hence state the conclusion of the hypothesis test in the case where $k = 24$ and $n = 42$. [1]

When $n = 42$, $H_0: \mu = 25$ is rejected in favour of $H_1: \mu \neq 25$ when $k \leq 23.793$ or $k \geq 26.207$. Since $k = 24$ does not satisfy the inequalities, we do not reject H_0 at 1% level of significant and conclude that there is insufficient evidence to suggest that $\mu \neq 25$.	

[Turn Over