Name:		Centre/Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/01

Paper 1

14 September 2022

3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**7**	**6**	**7**	**6**	**8**	**9**	**9**	**12**	**12**	**12**	**12**	**100**

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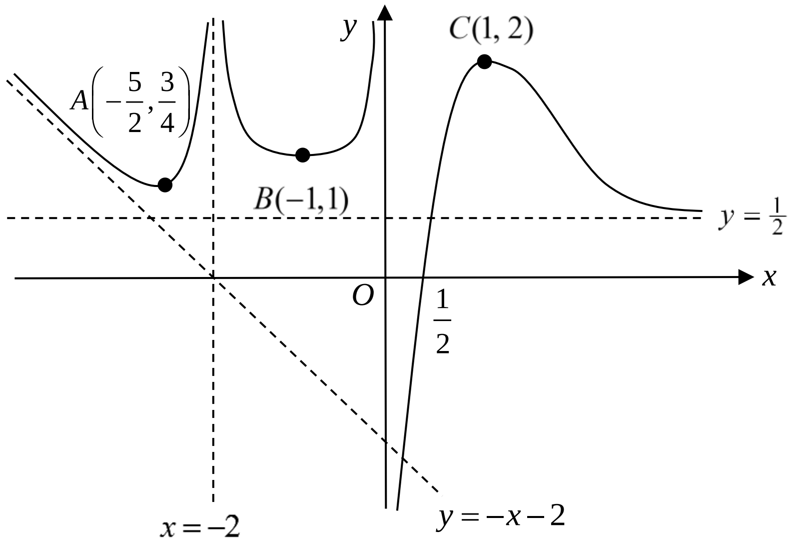
1 **(a)** Show that $-x^2+6x-14$ is always negative. [1]

(b) Without using a calculator, solve the inequality $\dfrac{x-5}{x-2} \geq \dfrac{3}{4-x}$, giving your answer in exact form. [3]

(c) Hence find the solution for the inequality $\dfrac{\ln x+5}{\ln x+2} \geq \dfrac{3}{4+\ln x}$. [3]

2 The diagram below shows the graph of $y=\mathrm{f}(x)$. The curve crosses the x-axis at $x=\frac{1}{2}$ and has turning points at $A\left(-\dfrac{5}{2},\dfrac{3}{4}\right)$, $B(-1,1)$ and $C(1,2)$. It has asymptotes $y=-x-2$, $y=\frac{1}{2}$, $x=0$ and $x=-2$.



On separate diagrams, sketch the following graphs. In each case, show clearly the equations of any asymptotes, coordinates of any points of intersection with both axes and the points corresponding to A, B and C.

(a) $y=\dfrac{1}{\mathrm{f}(x)}$ [3]

(b) $y=\mathrm{f}'(x)$ [3]

3 **(a)** Differentiate $\mathrm{e}^{\sin^2 2x}$ with respect to x. [2]

(b) Find $\displaystyle\int \frac{\mathrm{e}^{\sin^2 2x}\sin 4x}{\sqrt{1+\mathrm{e}^{\sin^2 2x}}}\,\mathrm{d}x$. [2]

(c) Find the exact value of $\int_0^{\frac{\pi}{4}} \mathrm{e}^{\sin^2 2x} \sin 4x\cos^2 2x\,\mathrm{d}x$. [3]

4 For this question, you may use the result $\left(\cos\theta + \mathrm{i}\sin\theta\right)^n = (\cos n\theta + \mathrm{i}\sin n\theta)$.

Find the exact value(s) of θ, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, given that

$$(\cos\theta + \mathrm{i}\sin\theta)^3 + (\cos\theta + \mathrm{i}\sin\theta)^5 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\,.$$ [6]

5 A curve C has equation $y^2 = 2\sin x + 2xy$, where $0 \le x \le 2\pi$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\cos x + y}{y - x}$. [2]

(b) Hence find the coordinates of the stationary points and use the second derivative test to determine its nature. [6]

6 Let $y = \cos\left[\ln(1+x)\right]$ for $|x| < 1$.

(a) Show that $(1+x)^2 \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 0$. By further differentiation of this result, show that the Maclaurin series for y is given by

$$1 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 - \tfrac{5}{12}x^4 + \ldots.$$ [6]

(b) By using the result in part **(a)**, deduce the Maclaurin series for $\sin\left[\ln(1+x)\right]$ in ascending powers of x up to and including x^3. [2]

(c) Hence state the tangent to the curve $y = \sin\left[\ln(1+x)\right]$ at $x = 0$. [1]

7 **(a)** **(i)** Find, in terms of n and x, an expression for $\displaystyle\sum_{r=0}^{n} \frac{(x+3)^r}{4^{r+1}}$. [2]

(ii) Give a reason why the infinite series $\displaystyle\sum_{r=0}^{\infty} \frac{(x+3)^r}{4^{r+1}}$ converges when $x = -5$ and determine its value. [2]

(b) **(i)** Given that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, find in terms of k, the sum $\displaystyle\sum_{r=6}^{2k} r(3r-2)$. [3]

(ii) Using your result in part **(b)(i)**, evaluate $\displaystyle\sum_{r=10}^{66} (r-4)(3r-14)$. [2]

8 The function f is defined as follows:

$$\mathrm{f}: x \mapsto \ln x^2 + 2x + 1 \quad \text{for } x \in \mathbb{R}, x \neq 0.$$

Another function g, defined for $x \in \mathbb{R}$, has the following properties:

- The function g is a continuous decreasing function with $\mathrm{g}(0) = -1$.
- The graph of $y = \mathrm{g}(x)$ has exactly one asymptote. The equation of this asymptote is $y = 0$.

(a) Sketch, on separate diagrams, the graph of $y = \mathrm{f}(x)$ and a possible graph for $y = \mathrm{g}(x)$. [4]

(b) Explain why the composite function fg exists and find the corresponding range of fg. [2]

(c) Given that $\mathrm{fg}(x) = 14x + 1 - 2\mathrm{e}^{7x}$, find an expression for $\mathrm{g}(x)$ in terms of x. [2]

(d) If the domain of f is further restricted to $x > k$, state the least value of k for which the function f^{-1} exists. Verify that $\mathrm{f}(1) = 3$ and use this result to find the gradient of the tangent to the curve $y = \mathrm{f}^{-1}(x)$ at $x = 3$. [4]

9 The plane p_1 contains the point $A(-4, 4, 0)$ and the line with equation $\dfrac{x-2}{2} = \dfrac{y-4}{3}$, $z = 6$.

The plane p_2 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, where λ and μ are real constants.

(a) Show that the line of intersection between p_1 and p_2 is parallel to the vector $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$.

Hence write down the vector equation of the line of intersection between p_1 and p_2. [5]

(b) Determine the acute angle between p_1 and p_2. [2]

(c) Find the position vector of the foot of perpendicular from point A to p_2. [3]

(d) The plane p_3 has equation $ax + 3y + 2z = b$, where $a, b \in \mathbb{R}$. Find the values of a and b such that all three planes have a common line of intersection. [2]

10 An object is moving from rest in a gas chamber and t seconds later, its velocity v metres per second satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 5 - 0.2v^2.$$

It is given that $\frac{\mathrm{d}v}{\mathrm{d}t} > 0$ and $v \geq 0$ for all values of t.

(a) Find t in terms of v, simplifying your answer. [5]

(b) Describe how the velocity of the object varies with time. [3]

(c) The displacement of the object, in metres, at any time m seconds is given by $\int_0^m v \, \mathrm{d}t$.

By evaluating this integral, show that the displacement of the object is $5\ln\left(\frac{\mathrm{e}^m + \mathrm{e}^{-m}}{2}\right)$. [4]

11 [It is given that the volume of a circular cone with base radius r and vertical height H is $\frac{1}{3}\pi r^2 H$.]

Intravenous (IV) fluids are specially formulated liquids that are injected into one of the veins of patients for medical treatment. The fluids are contained in a bag connected to an IV tube. The rate of flow of IV fluids into the body is regulated either manually or using an electric pump.

Figure 1 shows an IV fluid bag, modelled by a cylindrical and a conical section. The bag is initially filled with fluid. The radius of the cylinder and the cone is 4 cm. The heights of the cylindrical and conical sections are 20 cm and 4 cm respectively. The height from the apex of the cone to the fluid level is h cm.

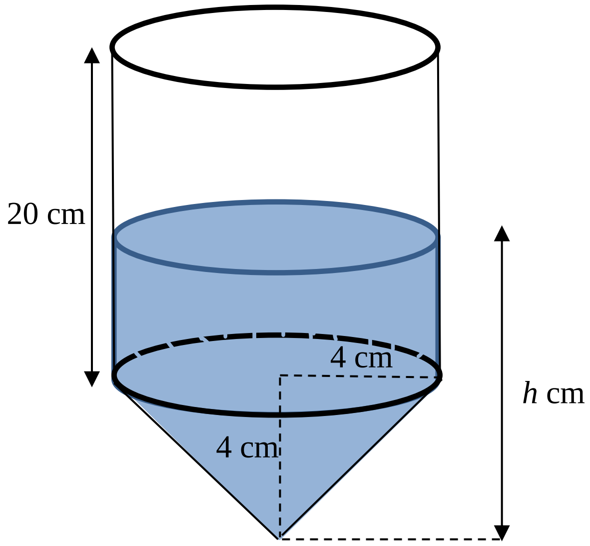


Figure 1

(a) Fluid is to be delivered to the patient at a rate of 50 cm^3 per hour. Determine the value of h and the volume of IV fluid left in the bag when h is decreasing at a rate of 1.21 cm per hour. [5]

Once the IV fluid enters the body, it is mixed with the blood and circulated through the blood vascular system (arteries, capillaries and veins). Due to the viscous nature of blood and friction with the blood vessel walls, resistance to blood flow is generated. To measure this resistance, the blood vessel is modelled as a cylinder with length L and radius r. The resistance R is given by the equation

$$R = \mu\left(\frac{L}{r^4}\right),$$

where μ is a positive constant determined by the viscosity of the blood in the body.

Figure 2 shows a main blood vessel with radius r_1, branching at an acute angle of θ into a smaller blood vessel with radius r_2. The length AD is given as m while the length CD is given as k.

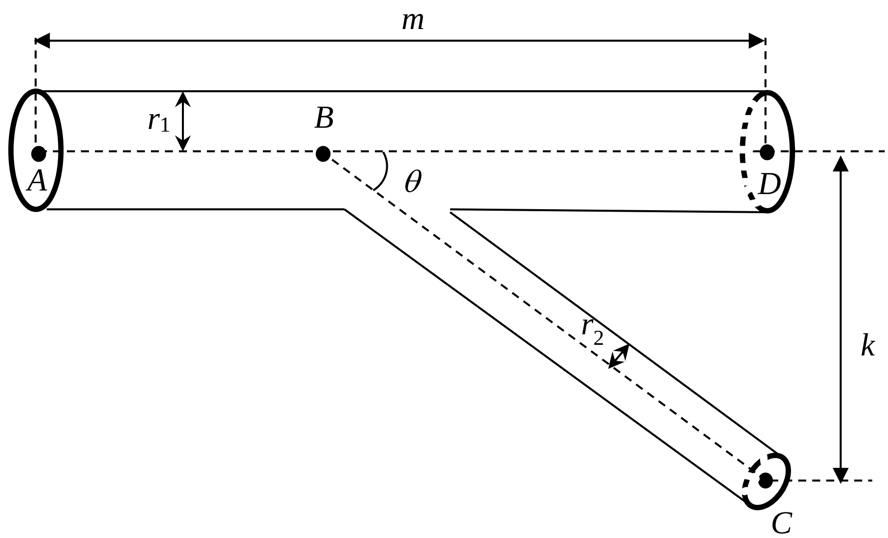


Figure 2

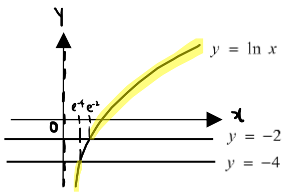
The total resistance, R_T, of the path ABC is the sum of the resistance of the path AB (with radius r_1) and the resistance of the path BC (with radius r_2).

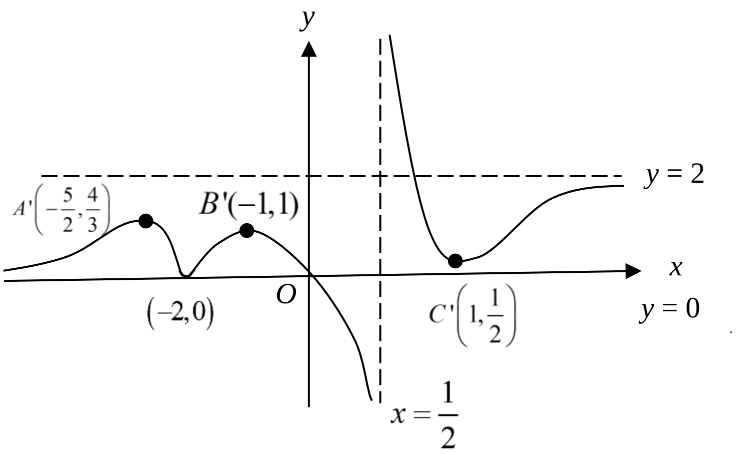
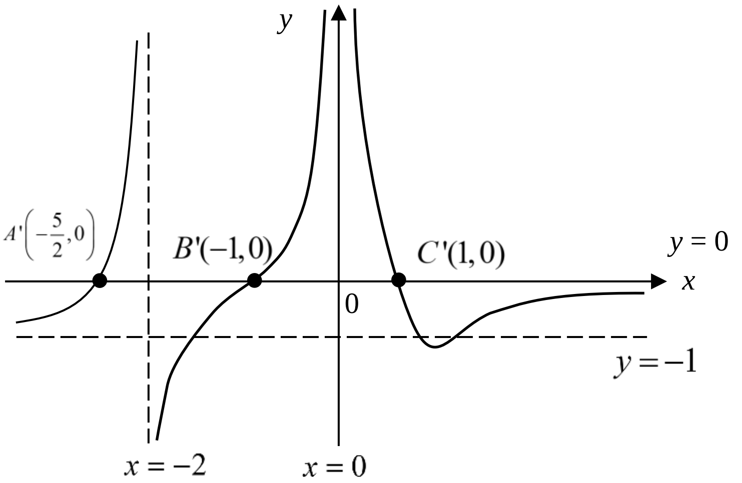
(b) Show that $R_T = \mu\left(\dfrac{m - k\cot\theta}{r_1^4} + \dfrac{k\operatorname{cosec}\theta}{r_2^4}\right)$. [2]

(c) As θ varies, show that $\cos\theta = \dfrac{r_2^4}{r_1^4}$ gives a stationary value of R_T. [3]

(d) Explain, with appropriate working, why the stationary value in part **(c)** provides the least resistance. [2]

DHS 2022 Year 6 H2 Math Prelim Exam P1 solutions

Qn	Suggested Solution
1(a)	$-x^2+6x-14=-\left(x^2-6x+14\right)$ $=-\left[\left(x-3\right)^2+5\right]$ $=-\left(x-3\right)^2-5<0$ for all real values of x $\left(\because \left(x-3\right)^2 \geq 0\right)$ **Alternative** Discriminant of $-x^2+6x-14=6^2-4\left(-1\right)\left(-14\right)$ $=36-56$ $=-20<0$ Since coeff of $x^2<0 \Rightarrow -x^2+6x-14<0$ for all real values of x
(b)	$\dfrac{x-5}{x-2} \geq \dfrac{3}{4-x}$ $\dfrac{\left(x-5\right)\left(4-x\right)-3\left(x-2\right)}{\left(x-2\right)\left(4-x\right)} \geq 0$ $\dfrac{-x^2+6x-14}{\left(x-2\right)\left(4-x\right)} \geq 0$ Since $-x^2+6x-14<0$ for all real x, $\left(x-2\right)\left(4-x\right)<0$ $x<2$ or $x>4$
(c)	$\dfrac{x-5}{x-2} > \dfrac{3}{4-x}$, Replace x with $-\ln x$. $\dfrac{\ln x+5}{\ln x+2} > \dfrac{3}{4+\ln x}$. $-\ln x<2$ or $-\ln x>4$ $\ln x>-2$ or $\ln x<-4$ $x>e^{-2}$ or $0<x<e^{-4}$ 

Qn	Suggested Solution
2(a)	
(b)	

Qn	Suggested Solution
3(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\sin^2 2x} = 4\mathrm{e}^{\sin^2 2x}\sin 2x\cos 2x = 2\mathrm{e}^{\sin^2 2x}\sin 4x$
(b)	$\int \frac{\mathrm{e}^{\sin^2 2x}\sin 4x}{\sqrt{1+\mathrm{e}^{\sin^2 2x}}}\,\mathrm{d}x$ $=\frac{1}{2}\int\left(2\mathrm{e}^{\sin^2 2x}\sin 4x\right)\left(1+\mathrm{e}^{\sin^2 2x}\right)^{-\frac{1}{2}}\,\mathrm{d}x$ $=\left(1+\mathrm{e}^{\sin^2 2x}\right)^{\frac{1}{2}}+c$ $=\sqrt{1+\mathrm{e}^{\sin^2 2x}}+c$
(c)	$\int_0^{\frac{\pi}{4}}\left(\mathrm{e}^{\sin^2 2x}\sin 4x\right)\cos^2 2x\,\mathrm{d}x$ $=\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\cos^2 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\frac{1}{2}\mathrm{e}^{\sin^2 2x}\left(-4\cos 2x\sin 2x\right)\,\mathrm{d}x$ $=-\frac{1}{2}+\int_0^{\frac{\pi}{4}}\mathrm{e}^{\sin^2 2x}\sin 4x\,\mathrm{d}x$ $=-\frac{1}{2}+\left[\frac{1}{2}\mathrm{e}^{\sin^2 2x}\right]_0^{\frac{\pi}{4}}$ $=-\frac{1}{2}+\frac{1}{2}\mathrm{e}-\frac{1}{2}$ $=\frac{1}{2}\mathrm{e}-1$

Qn	Suggested Solution
4	**<u>Method 1</u>** $(\cos\theta+\mathrm{i}\sin\theta)^3+(\cos\theta+\mathrm{i}\sin\theta)^5=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $(\cos 3\theta+\mathrm{i}\sin 3\theta)+(\cos 5\theta+\mathrm{i}\sin 5\theta)=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $(\cos 3\theta+\cos 5\theta)+\mathrm{i}(\sin 3\theta+\sin 5\theta)=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $2\cos 4\theta\cos\theta+2\mathrm{i}\sin 4\theta\cos\theta=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $2\cos\theta[\cos 4\theta+\mathrm{i}\sin 4\theta]=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $[2\cos\theta]\mathrm{e}^{\mathrm{i}(4\theta)}=\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}$ $\left\|(2\cos\theta)\mathrm{e}^{\mathrm{i}(4\theta)}\right\|=\left\|\mathrm{e}^{\mathrm{i}(-\frac{2\pi}{3})}\right\|=1$ $\|2\cos\theta\|=1 \;\Rightarrow\; \theta=\pm\frac{\pi}{3}$ Comparing the argument of 4θ with $-\frac{2\pi}{3}$, only $\theta=\frac{\pi}{3}$ is valid. **<u>Method 2</u>** $(\cos\theta+\mathrm{i}\sin\theta)^3+(\cos\theta+\mathrm{i}\sin\theta)^5=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ $(\cos 3\theta+\cos 5\theta)+\mathrm{i}(\sin 3\theta+\sin 5\theta)=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ $2\cos 4\theta\cos\theta+2\mathrm{i}\sin 4\theta\cos\theta=-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ Comparing real and imaginary parts, $2\cos 4\theta\cos\theta=-\frac{1}{2}$ (1) $2\sin 4\theta\cos\theta=-\frac{\sqrt{3}}{2}$ (2) $\frac{(2)}{(1)}$, $\tan 4\theta=\sqrt{3}$ $4\theta=-\frac{5\pi}{3},-\frac{2\pi}{3},\frac{\pi}{3},\frac{4\pi}{3}$ $\theta=-\frac{5\pi}{12},-\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{3}$ Only $\theta=\frac{\pi}{3}$ satisfies (1) and (2).

Qn	Suggested Solution
5(a)	$y^2 = 2\sin x + 2xy$ ... (1) Differentiate with respect to x, $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x + 2y$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x + y}{y - x}$ (shown)
(b)	For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x + y}{y - x} = 0$ $\cos x + y = 0$ $y = -\cos x$ Substitute $y = -\cos x$ into (1) $(-\cos x)^2 = 2\sin x + 2x(-\cos x)$ $\cos^2 x - 2\sin x + 2x\cos x = 0$ From GC : $x = 0.87394$ or 4.4877 $y = -\cos(0.87394)$ or $-\cos(4.4877)$ $= -0.64181$ or 0.22280 Coordinates : $P(0.874, -0.642)$ & $Q(4.49, 0.223)$ $(y - x)\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x + y$ $(y - x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = -\sin x + \frac{\mathrm{d}y}{\mathrm{d}x}$ When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $(y - x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\sin x$ At $x = 0.87394$ and $y = -0.64181$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0.50593 > 0$ $\therefore (0.874, -0.642)$ is a minimum point. At $x = 4.4877$ and $y = 0.22280$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -0.22858 < 0$ $\therefore (4.49, 0.223)$ is a maximum point.

Qn	Suggested Solution
6(a)	Method 1 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sin\left[\ln(1+x)\right]}{1+x}$ $\therefore (1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin\left[\ln(1+x)\right]$ Differentiating with respect to x, $(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\cos\left[\ln(1+x)\right]}{1+x}=-\frac{y}{1+x}$ $\therefore (1+x)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}+y=0$ (shown) Method 2 $\cos^{-1}y=\ln(1+x)$ Differentiating with respect to x, $-\frac{1}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{1}{1+x}$ $\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{1-y^2}$ Differentiating with respect to x, $\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\frac{1}{2}\left(-2y\right)}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{y}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $\left(1+x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y}{1+x}\quad\because \frac{1}{\sqrt{1-y^2}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=-\frac{1}{1+x}$ $\left(1+x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-y$ $\left(1+x\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+y$ (shown)
	Differentiating with respect to x, $(1+x)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\therefore (1+x)^2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3(1+x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}=0$ Differentiating again with respect to x, $(1+x)^2\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+2(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+3\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ $\therefore (1+x)^2\frac{\mathrm{d}^4y}{\mathrm{d}x^4}+5(1+x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+5\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ When $x=0,\ \ y=1,\frac{\mathrm{d}y}{\mathrm{d}x}=0,\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1,\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=3,\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-10$ $\therefore y=1-\frac{1}{2!}x^2+\frac{3}{3!}x^3-\frac{10}{4!}x^4+\ldots$ $\quad=1-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{12}x^4+\ldots$ (shown)

(b)	$\sin\left[\ln(1+x)\right] = -(1+x)\dfrac{dy}{dx}$ $= -(1+x)\left(-x+\tfrac{3}{2}x^2-\tfrac{5}{3}x^3+\ldots\right)$ $= x-\tfrac{1}{2}x^2+\tfrac{1}{6}x^3+\ldots$
(c)	$\therefore$ Equation of tangent to curve at $x=0$ is $y=x$.

	Suggested Solution
7(a)(i)	$\displaystyle\sum_{r=0}^{n}\frac{(x+3)^r}{4^{r+1}} = \frac{1}{4}\sum_{r=0}^{n}\left(\frac{x+3}{4}\right)^r$ $\displaystyle= \frac{1}{4}\sum_{r=0}^{n}\left[\left(\frac{x+3}{4}\right)^0+\left(\frac{x+3}{4}\right)^1+\ldots+\left(\frac{x+3}{4}\right)^n\right]$ $\displaystyle= \frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{1-\frac{x+3}{4}}\right]$ $\displaystyle= \frac{1}{4}\left[\frac{1-\left(\frac{x+3}{4}\right)^{n+1}}{\frac{1-x}{4}}\right]$ $\displaystyle= \frac{1}{1-x}\left[1-\left(\tfrac{x+3}{4}\right)^{n+1}\right]$
(ii)	Common ratio, r of G.P. = $\dfrac{x+3}{4}$ When $x=-5$, $r=\dfrac{-5+3}{4}=-\dfrac{1}{2}$. Since $\lvert r\rvert=\dfrac{1}{2}<1$, the G.P. converges. Hence, the series $\displaystyle\sum_{r=0}^{n}\frac{(x+3)^r}{4^{r+1}}$ converges. $\displaystyle\lim_{n\to\infty}\sum_{r=0}^{n}\frac{(-5+3)^r}{4^{r+1}} = \lim_{n\to\infty}\frac{1}{1-(-5)}\left[1-\left(\tfrac{-5+3}{4}\right)^{n+1}\right]$ $\displaystyle= \lim_{n\to\infty}\frac{1}{6}\left[1-\left(-\tfrac{1}{2}\right)^{n+1}\right]$ $\displaystyle= \frac{1}{6}$

<table>
<tr><td>(b)(i)</td><td>$\sum_{r=6}^{2k} r(3r-2) = \sum_{r=6}^{2k}(3r^2-2r)$
$= 3\left[\sum_{r=1}^{2k} r^2 - \sum_{r=1}^{5} r^2\right] - 2\sum_{r=6}^{2k} r$
$= 3\left[\frac{2k}{6}(2k+1)(4k+1) - \frac{5}{6}(6)(11)\right]$
$-2\left(\frac{2k-6+1}{2}(6+2k)\right)$
$= k(2k+1)(4k+1) - 165 - 2(2k-5)(3+k)$
$= k(2k+1)(4k+1) - 2(2k-5)(k+3) - 165$</td></tr>
<tr><td>(ii)</td><td><u>Method 1</u>
Replace r with $r + 4$ in $\sum_{r=10}^{66}(r-4)(3r-14)$.
$\sum_{r+4=10}^{r+4=66}\left((r+4)-4\right)\left(3(r+4)-14\right) = \sum_{r=6}^{r=62} r(3r-2)$

Compared with (b)(i), observe that $k = 31$.
$\sum_{r=10}^{66}(r-4)(3r-14)$
$= \sum_{r=6}^{r=62} r(3r-2)$
$= 31(2(31)+1)(4(31)+1) - 2(2(31)-5)((31)+3) - 165$
$= 240084$

<u>Method 2</u>
Replace r with $r - 4$ in $\sum_{r=6}^{2k} r(3r-2)$.
$\sum_{r-4=6}^{r-4=2k}(r-4)(3(r-4)-2) = k(2k+1)(4k+1) - 2(2k-5)(k+3) - 165$
$\sum_{r=10}^{2k+4}(r-4)(3r-14) = k(2k+1)(4k+1) - 2(2k-5)(k+3) - 165$
Observe that $2k+4 = 66 \Rightarrow k = 31$.
$\sum_{r=10}^{66}(r-4)(3r-14)$
$= 31(2(31)+1)(4(31)+1) - 2(2(31)-5)((31)+3) - 165$
$= 240084$</td></tr>
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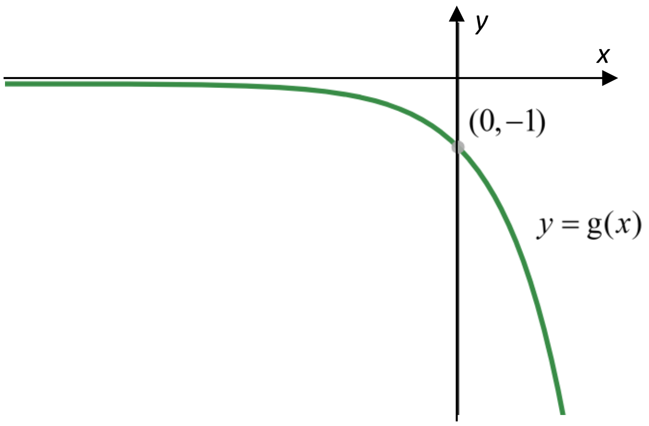
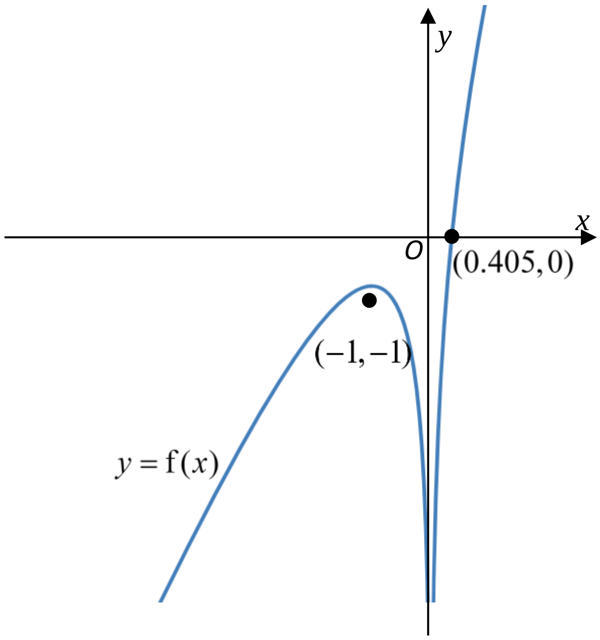
Qn	Suggested Solution
8(a)	
(b)	For composite function fg to exists, $R_g \subseteq D_f$. Since $(-\infty, 0) \subseteq (-\infty, 0) \cup (0, \infty)$, fg exists. $\mathbb{R} \xrightarrow{g} (-\infty, 0) \xrightarrow{f} (-\infty, -1]$. $R_{fg} = (-\infty, -1]$.

(c)	Let m = g(x), $\mathrm{fg}(x)=14x+1-2\mathrm{e}^{7x}$ $\mathrm{f}(m)=14x+1-2\mathrm{e}^{7x}$ $\ln(m^2)+2(m)+1=14x+1-2\mathrm{e}^{7x}$ $\ln(m^2)+2(m)=14x-2\mathrm{e}^{7x}$ Guess $2m=-2\mathrm{e}^{7x}$ $m=-\mathrm{e}^{7x}$ Verify $\ln(m^2)$ $=\ln\left[(-e^{7x})^2\right]$ $=\ln\left(\mathrm{e}^{14x}\right)$ $=14x$ $\therefore\ \mathrm{g}(x)=-\mathrm{e}^{7x}$.
(d)	From the graph, the least value of k is 0 . $\mathrm{f}(1)=\ln 1^2+2+1=0+2+1=3$ (verified) Gradient of $y=\mathrm{f}^{-1}(x)$ at (x = 3) = 1/(Gradient of $y=\mathrm{f}(x)$) at (x = 1) $=\dfrac{1}{\mathrm{f}'(1)}=0.250$

Qn	Suggested Solutions
9(a)	To find eqn of the line, $\begin{cases} \alpha = \dfrac{x-2}{2} \Rightarrow x = 2+2\alpha \\ \alpha = \dfrac{y-4}{3} \Rightarrow y = 4+3\alpha \\ \qquad z = 6 \end{cases} \Rightarrow \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \alpha \in \mathbb{R}$ Normal of plane p_1 $= \left(\begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \right) \times \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 18 \\ -12 \\ -18 \end{pmatrix} = 6 \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ Let $\mathbf{n_1} = \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}$ Normal of p_2, $\mathbf{n_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ **<u>Method 1</u>** For direction vector of line of intersection, $\mathbf{n_2} \times \mathbf{n_1} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$ Hence the line of intersection is parallel to $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$. (shown) Vector equation of the line of intersection is $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \beta \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \beta \in \mathbb{R}$ **<u>Method 2</u>** $p_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix} = -12-8 = -20 \Rightarrow 3x-2y-3z = -20$ $p_2 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2-4+6 = 4 \Rightarrow x-y+z = 4$

	Solve p_1 and p_2 using GC: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -28+5z \\ -32+6z \\ z \end{pmatrix} = \begin{pmatrix} -28 \\ -32 \\ 0 \end{pmatrix} + z\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$ Let $\beta = z$, we obtain the equation of the line of intersection as $\mathbf{r} = \begin{pmatrix} -28 \\ -32 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}, \ \beta \in \mathbb{R}$. Hence the line of intersection is parallel to $\begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix}$. (shown)
(b)	Acute angle between p_1 and p_2 $= \cos^{-1}\left\lvert \dfrac{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -2 \\ -3 \end{pmatrix}}{\sqrt{3}\sqrt{22}} \right\rvert = \cos^{-1}\left\lvert \dfrac{2}{\sqrt{3}\sqrt{22}} \right\rvert = 75.7^o$
(c)	From Q9(a) **Method 2**, $p_2 : \mathbf{r} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$ Let the foot of perpendicular from *A* to p_2 be *F*. $l_{AF} : \mathbf{r} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \ \gamma \in \mathbb{R}$ Since *F* lies on l_{AF}, $\overrightarrow{OF} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + \gamma\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ for some values of γ

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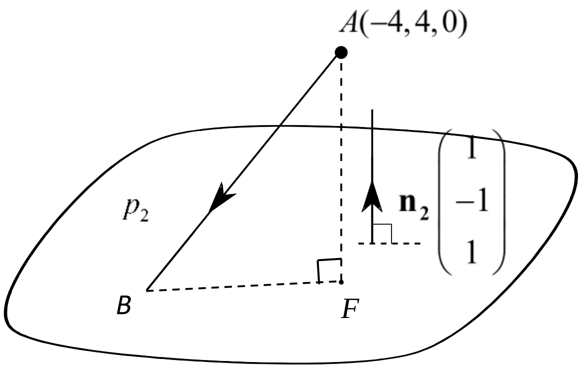
Since F also lies on p_2,



$$\begin{pmatrix} -4+\gamma \\ 4-\gamma \\ \gamma \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 4$$

$$-4+\gamma-4+\gamma+\gamma=4$$

$$3\gamma=12 \Rightarrow \gamma=4$$

$$\overrightarrow{OF} = \begin{pmatrix} -4 \\ 4 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$
</td></tr>
<tr><td>(d)</td><td>
Since all three planes have a common line of intersection, the point (2, 4, 6) must also lie on p_3. Substitute (2, 4, 6) into $ax+3y+2z=b$.

$$\begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} = b \Rightarrow 2a+24=b$$

and the normal of p_3 must be perpendicular to the line of intersection,

$$\begin{pmatrix} a \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 6 \\ 1 \end{pmatrix} = 0 \Rightarrow 5a+18+2=0 \Rightarrow a=-4$$

Therefore $a=-4$, $b=16$.
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Qn	Suggested Solution
10(a)	$\dfrac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2$ $\displaystyle\int \frac{1}{5-0.2v^2}\,\mathrm{d}v=\int \mathrm{d}t$ $\displaystyle 5\int \frac{1}{25-v^2}\,\mathrm{d}v=\int \mathrm{d}t$ $5\left[\dfrac{1}{2(5)}\ln\left\|\dfrac{5+v}{5-v}\right\|\right]+C=t$ Since $\dfrac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2>0$ $\Rightarrow 0\le v<5 \quad \text{as } v\ge 0$ $5\left[\dfrac{1}{2(5)}\ln\left(\dfrac{5+v}{5-v}\right)\right]+C=t$, where C is an arbitrary constant $\therefore\ t=\dfrac{1}{2}\ln\left(\dfrac{5+v}{5-v}\right)+C$ Since at $t=0$, $v=0$; $0=\dfrac{1}{2}\ln\left(\dfrac{5+v}{5-v}\right)+C$ $\Rightarrow C=0$ $\therefore t=\dfrac{1}{2}\ln\left(\dfrac{5+v}{5-v}\right)$
(b)	Since $\dfrac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2$, The velocity of the object is increasing at a decreasing rate. **<u>Method 1</u>** $t=\dfrac{1}{2}\ln\left(\dfrac{5+v}{5-v}\right)\Rightarrow\ v=\dfrac{5(\mathrm{e}^{2t}-1)}{\mathrm{e}^{2t}+1}=\dfrac{5(1-\mathrm{e}^{-2t})}{1+\mathrm{e}^{-2t}}$ As $t\to\infty$, $v\to 5$. In the long run, the velocity of the object will travel at 5 m/s. **<u>Method 2</u>** $t=\dfrac{1}{2}\ln\left(\dfrac{5+v}{5-v}\right)\Rightarrow \mathrm{e}^{2t}=\dfrac{5+v}{5-v}$ As $t\to\infty\Rightarrow 5-v\to 0\Rightarrow v\to 5$

(c)	$\int_0^m v \, \mathrm{d}t$ $= \int_0^m \frac{5(\mathrm{e}^{2t}-1)}{(\mathrm{e}^{2t}+1)} \, \mathrm{d}t$ $= 5\left(\int_0^m \frac{(\mathrm{e}^{2t}+1)-2}{\mathrm{e}^{2t}+1} \, \mathrm{d}t \right)$ $= 5\left(\int_0^m 1 - \frac{2}{\mathrm{e}^{2t}+1} \, \mathrm{d}t \right)$ $= 5\left(\int_0^m 1 - \frac{2\mathrm{e}^{-2t}}{\mathrm{e}^{-2t}\left(\mathrm{e}^{2t}+1\right)} \, \mathrm{d}t \right)$ $= 5\left(\int_0^m 1 - \frac{2\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t \right)$ $= 5\left[t + \ln(1+\mathrm{e}^{-2t}) \right]_0^m$ $= 5\left[m + \ln(1+e^{-2m}) - \ln(2) \right]$ $= 5\left[\ln(e^m) + \ln(1+e^{-2m}) - \ln(2) \right]$ $= 5\left[\ln\left(\frac{e^m(1+e^{-2m})}{2} \right) \right]$ $= 5\ln\left(\frac{e^m + e^{-m}}{2} \right)$ (shown) **Alternative method:** $\int_0^m v \, \mathrm{d}t$ $= \int_0^m \frac{5(\mathrm{e}^{2t}-1)}{(\mathrm{e}^{2t}+1)} \, \mathrm{d}t$ $= 5\left(\int_0^m \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{1}{\mathrm{e}^{2t}+1} \, \mathrm{d}t \right)$ $= 5\left(\int_0^m \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{\mathrm{e}^{-2t}}{\mathrm{e}^{-2t}\left(\mathrm{e}^{2t}+1\right)} \, \mathrm{d}t \right)$ $= 5\left(\frac{1}{2}\int_0^m \frac{2\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} \, \mathrm{d}t - \int_0^m \frac{\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t \right)$ $= 5\left[\frac{1}{2}\ln\left(\mathrm{e}^{2t}+1\right) \right]_0^m + \frac{5}{2}\int_0^m \frac{-2\mathrm{e}^{-2t}}{1+\mathrm{e}^{-2t}} \, \mathrm{d}t$ $= 5\left[\frac{1}{2}\ln\left(\mathrm{e}^{2t}+1\right) + \frac{1}{2}\ln(1+\mathrm{e}^{-2t}) \right]_0^m$

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$$= \frac{5}{2}\left[\ln\left(\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}\right)\right]_0^m$$

$$= \frac{5}{2}\ln\left(\frac{\mathrm{e}^{2m}+2+\mathrm{e}^{-2m}}{4}\right)$$

$$= \frac{5}{2}\ln\left(\frac{\left(\mathrm{e}^{m}+\mathrm{e}^{-m}\right)^2}{2^2}\right)$$

$$= 5\ln\left(\frac{e^{m}+e^{-m}}{2}\right) \text{ (shown) } \because \frac{e^{m}+e^{-m}}{2} > 0$$

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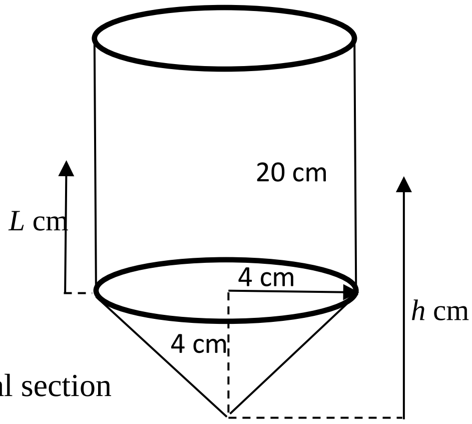
The liquid level could either be in the cylindrical or conical section. Assume it’s in the cylindrical section of height L and constant cross-sectional area of 16π



$V = \pi r^2 L = 16\pi L$

$\frac{\mathrm{d}V}{\mathrm{d}t} = (16\pi)\frac{\mathrm{d}L}{\mathrm{d}t}$

$\Rightarrow \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{-50}{16\pi}$

$= -0.99472$ cm/h for $h \geq 4$

Since $\frac{\mathrm{d}L}{\mathrm{d}t} = -0.99472 \neq -1.21$

$\Rightarrow$ liquid level is in the conical section

$\Rightarrow h < 4$

Let r be radius of water surface in conical section.

From diagram, $r = h$.

$V = \frac{1}{3}\pi(r)^2 h = \frac{1}{3}\pi h^3$

$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi h^2$

$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}$

$-50 = \pi h^2(-1.21)$

$h = 3.6267$

$h = 3.63$ cm (3 sf)

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	$V=\frac{1}{3}\pi\left(3.6267\right)^3$ $=49.9552$ $=50.0\text{ cm}^3$ (3 sf)
(b)	$\dfrac{k}{BD}=\tan\theta \Rightarrow BD=k\cot\theta$ $AB=m-k\cot\theta$ $R_{AB}=\mu\left(\dfrac{m-k\cot\theta}{r_1^4}\right)$ $\dfrac{k}{BC}=\sin\theta \Rightarrow BC=k\text{cosec}\theta$ $R_{BC}=\mu\left(\dfrac{k\text{cosec}\theta}{r_2^4}\right)$ Therefore, $R_T=\mu\left(\dfrac{m-k\cot\theta}{r_1^4}+\dfrac{k\text{cosec}\theta}{r_2^4}\right)$ (shown)
(c)	$\dfrac{\mathrm{d}R_T}{\mathrm{d}\theta}=\mu\left[-\dfrac{k\left(-\text{cosec}^2\theta\right)}{r_1^4}+\dfrac{k\left(-\text{cosec}\theta\cot\theta\right)}{r_2^4}\right]$ $=\mu k\left(\dfrac{\text{cosec}^2\theta}{r_1^4}-\dfrac{\text{cosec}\theta\cot\theta}{r_2^4}\right)$ $=\mu k\left(\dfrac{\text{cosec}^2\theta}{r_1^4}-\dfrac{\text{cosec}^2\theta\cos\theta}{r_2^4}\right)$ $=\mu k\text{cosec}^2\theta\left(\dfrac{1}{r_1^4}-\dfrac{\cos\theta}{r_2^4}\right)$ When $\dfrac{\mathrm{d}R_T}{\mathrm{d}\theta}=0$, $\mu k\text{cosec}^2\theta\left(\dfrac{1}{r_1^4}-\dfrac{\cos\theta}{r_2^4}\right)=0$ $\text{cosec}\theta=0$ (no soln.) or $\dfrac{1}{r_1^4}-\dfrac{\cos\theta}{r_2^4}=0$ $\cos\theta=\dfrac{r_2^4}{r_1^4}$ (shown)

(d)	**Method 1**

$$\frac{dR_T}{d\theta} = \mu k \operatorname{cosec}^2\theta\left(\frac{1}{r_1^4} - \frac{\cos\theta}{r_2^4}\right)$$

$$= \frac{\mu k \operatorname{cosec}^2\theta}{r_1^4 r_2^4}\left(r_2^4 - r_1^4\cos\theta\right)$$

$$= \frac{\mu k \operatorname{cosec}^2\theta}{r_2^4}\left(\frac{r_2^4}{r_1^4} - \cos\theta\right)$$

As $\cos\theta^- > \cos\theta > \cos\theta^+$ when θ is acute,

θ^-	θ	θ^+
$\cos\theta^- > \frac{r_2^4}{r_1^4}$	$\cos\theta = \frac{r_2^4}{r_1^4}$	$\cos\theta^+ < \frac{r_2^4}{r_1^4}$
$\frac{dR_T}{d\theta} < 0$	$\frac{dR_T}{d\theta} = 0$	$\frac{dR_T}{d\theta} > 0$

$\therefore$ Stationary value in part (c) is a minimum i.e. least resistance.

Method 2

$$\frac{d^2R_T}{d\theta^2} = \mu\left[-\frac{k\left(2\operatorname{cosec}\theta\right)\left(-\operatorname{cosec}\theta\cot\theta\right)}{r_1^4} - \frac{k\left(-\operatorname{cosec}\theta\cot^2\theta + \operatorname{cosec}\theta\left(-\operatorname{cosec}^2\theta\right)\right)}{r_2^4}\right]$$

$$= \mu k\left[-\frac{2\operatorname{cosec}^2\theta\cot\theta}{r_1^4} + \frac{\operatorname{cosec}\theta\cot^2\theta + \operatorname{cosec}^3\theta}{r_2^4}\right]$$

$$= \mu k\left[-\frac{2}{r_1^4}\left(\frac{1}{\sin^2\theta}\cdot\frac{\cos\theta}{\sin\theta}\right) + \frac{1}{r_2^4}\left(\frac{1}{\sin\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta} + \frac{1}{\sin^3\theta}\right)\right]$$

$$= \frac{\mu k}{\sin^3\theta}\left[\frac{1}{r_2^4}\left(1+\cos^2\theta\right) - \frac{2}{r_1^4}\cos\theta\right]$$

When θ is acute and $\cos\theta = \frac{r_2^4}{r_1^4}$, $\frac{\mu k}{\sin^3\theta} > 0$ and

	$$\frac{1}{r_2^4}\left(1+\cos^2\theta\right)-\frac{2}{r_1^4}\cos\theta=\frac{1}{r_2^4}\left(1+\frac{r_2^8}{r_1^8}\right)-\frac{2}{r_1^4}\cdot\frac{r_2^4}{r_1^4}$$ $$=\frac{1}{r_2^4}+\frac{r_2^4}{r_1^8}-2\frac{r_2^4}{r_1^8}$$ $$=\frac{1}{r_2^4}-\frac{r_2^4}{r_1^8}$$ $$=\frac{r_1^8-r_2^8}{r_1^8r_2^4}$$ $$>0 \text{ (since } r_1>r_2\Rightarrow r_1^8>r_2^8\text{)}$$ Hence, $\frac{\mathrm{d}^2R_T}{\mathrm{d}\theta^2}>0\Rightarrow$ Minimum value.

Name:		Centre/Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) — 9758/02

Paper 2

19 September 2022

3 hours

Candidates answer on the Question Paper

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**5**	**6**	**7**	**9**	**13**	**7**	**9**	**10**	**9**	**12**	**13**	**100**

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Section A: Pure Mathematics [40 marks]

1 **(a)** Sketch the curve C given by the equation

$$4x^2 - 9(y+1)^2 - 36 = 0,$$

indicating clearly the equations of any asymptotes. [2]

(b) Find the volume generated when the region bounded by C and the line $x = -\frac{\sqrt{45}}{2}$ is rotated through 2π radians about the y-axis. [3]

2 **Do not use a calculator in answering this question.**

The complex number w is given by $2-3\mathrm{i}$.

(a) Find w^3 in the form $x+\mathrm{i}y$, showing your working. [2]

w is a root of the equation $z^3 - 5z^2 + az + b = 0$ where a and b are real constants. This equation also has a real root α.

(b) Find the values of a and b. [2]

(c) Find a cubic polynomial that has roots $\mathrm{i}w, \mathrm{i}w^*$ and $\mathrm{i}\alpha$ in the form $z^3 + pz^2 + qz + r = 0$, where p, q and r are real constants. [2]

3 The curve C has equation $y = \dfrac{4(x-2)^2}{x-4}$.

(a) Sketch C. Label the coordinates of any stationary points and point(s) of intersection with the axes. State the equations of its asymptotes and the coordinates of the point of intersection.[3]

(b) Deduce the range of values of a such that the equation

$$(x-4)^2 + \left(\frac{4(x-2)^2}{x-4} - 16\right)^2 = a$$

has a negative real root. [2]

(c) Suppose the variable point P on C represents the complex number $z = x + \mathrm{i}\,y$. By referring to the point of intersection between the asymptotes of C, find the range of values of $\arg(z - 4 - 16\mathrm{i})$. [2]

4 Relative to the origin O, the position vectors of the points P, Q and R are $\mathbf{i}$, $2\mathbf{j} - t\mathbf{k}$ and $t\mathbf{k}$ respectively, where t is a fixed constant. The points A and B divides both line segments PQ and QR respectively in the same ratio of $\mu : 1 - \mu$, where μ is a parameter such that $0 < \mu < 1$.

(a) Find the vector $\overrightarrow{AB}$ in terms of t and μ. [3]

(b) Determine whether the points O, A, B are collinear. [1]

(c) Find the values of μ such that the length of projection of $\overrightarrow{AB}$ onto $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ is $\frac{1}{5}$ unit. [2]

(d) Given that angle AOB is a right angle, find the set of possible values of t, justifying your answer clearly. [3]

5 **(a)** Show, by integration, that $\int \sin x(1-\sin x)\,\mathrm{d}x = -\cos x - \dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + D$ where D is an arbitrary constant. [2]

With reference from the origin O, the curve C has parametric equations

$$x = \theta(1-\sin\theta),\ y = 1-\cos\theta, \quad \text{for } 0 \le \theta \le \tfrac{\pi}{2}.$$

The line L has the equation $y = 2x$.

(b) Sketch C, showing clearly the coordinates of any point(s) of intersection with the axes. [2]

(c) A point P lies on C where the tangent at P is parallel to L. Find the coordinates of P. [3]

(d) Find the exact area of the region enclosed by C and the y-axis. [4]

(e) A point Q lies on C such that OQ makes an angle of $\frac{\pi}{6}$ radians with L. Find the value of the parameter θ for Q. [2]

Section B: Probability and Statistics [60 marks]

6 **(a)** Find the number of ways the letters of the word ABDUCTIONS can be arranged in a circle such that the letters A, O and U must not be next to one another. [2]

(b) A five-letter codeword is randomly formed from the word TYRANNOSAUR.

(i) Find the number of codewords that can be formed if it contains at most 1 pair of identical letters. [3]

(ii) Find the probability that a codeword contains both R's and both N's, given that it contains the block of letters "RAN". [2]

7 Commodity X is **traded four times a week from Monday to Thursday**. The unit price of X can only rise or fall on any day. If the unit price of X rises on a day, there is a probability of 0.6 that it will rise on the next trading day. If the unit price of X falls on a day, there is a probability of 0.15 that it will rise on the next trading day.

In a particular week, the unit price of X rises on Monday and the events A and B are defined as follows.

A: the unit price of X falls on Tuesday.

B: the unit price of X rises on Thursday.

(a) Find

(i) $\mathrm{P}(A \cap B)$, [2]

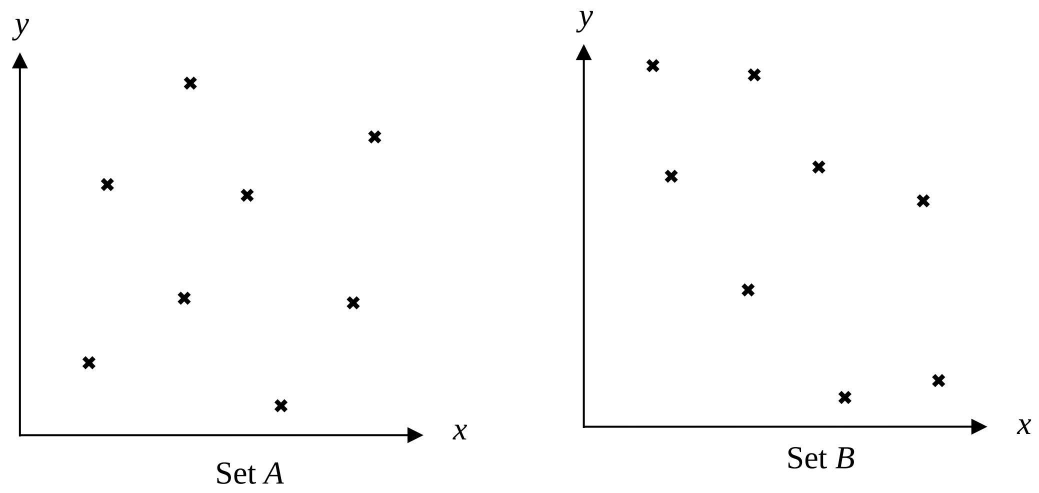
(ii) $\mathrm{P}(B)$, [2]

(iii) $\mathrm{P}(B \mid A)$. [2]

(b) State, with a reason, whether A and B are independent. [1]

(c) If the unit price of X rises for 12 consecutive Mondays in a particular year, find the probability that it rises on exactly 5 of the corresponding Tuesdays. [2]

8 **(a)** The scatter diagrams of two different sets of data points are shown as follows:



Determine, with a reason, which data set would result in a larger absolute value of the product moment correlation coefficient. [2]

(b) A researcher investigates the relationship between the yearly mean household income, x thousands, and the yearly mean household expenditure, y thousands. The table below shows the detailed data over 10 years.

x	52.0	54.1	48.8	49.1	45.5	47.9	51.0	50.4	53.5	55.5
y	47.0	53.1	46.0	45.6	45.5	45.0	46.3	46.1	48.0	54.0

(i) Give a sketch of the scatter diagram of the data. [1]

(ii) The researcher proposes two models to represent the data:

$$C:\ y = a + b\ln x$$

$$D:\ y = a + be^{\frac{1}{2}x}$$

By calculating the product moment correlation coefficients for this data, explain which is the more appropriate model. [2]

Use the more appropriate model in part **(ii)** for the rest of the question.

(iii) Use a regression line to obtain the estimate for the household expenditure for a household income of \$50 000, correct to the nearest integer. Comment on the reliability of this estimate. [3]

(iv) Explain if it is valid to conclude that a higher income will result in a higher expenditure. [1]

(v) A student claims that the product moment correlation coefficient will remain the same between the models $y = a + be^{\frac{1}{2}x}$ and $y = p + qe^{\frac{1}{20}x}$. Comment whether the claim is true. [1]

9 **(a)** A confectionery is selling mooncakes during the Mid-Autumn Festival. The owner claims that the mean mass of the mooncakes is at least 150 g. The distribution of the mass of a mooncake has standard deviation 6.73 g. A customer bought a random selection of 9 mooncakes from the confectionery with masses, in g, as follows

145 148 153 156 141 151 143 157 138

Test at the 10% significance level whether the owner's claim is valid. State an assumption about the population distribution of the mass of the mooncakes. [5]

(b) In a particular country, the mean working hours of teachers in schools is 60 hours per week. After implementation of the Home-Based Learning in all schools, the Ministry claims that the mean working hours of teachers remained unchanged. A random sample of 50 teachers was taken and the number of working hours per week was recorded. The mean working hours and standard deviation for the sample were found to be 62 hours and k hours respectively. A hypothesis test was conducted and it was found that there is sufficient evidence to reject the Ministry's claim at 5% significance level. Find the set of values of k. [4]

10 In this question you should state the parameters of any distributions that you use.

A fruit stall vendor sells rock melons and watermelons. The masses (in grams) of a randomly chosen rock melon and watermelon, denoted by X and Y respectively, have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean (g)	Standard deviation (g)
X	580	22
Y	870	30

(a) Sketch the distribution of X for masses between 558 g to 646 g. [1]

(b) Find the expected number of rock melons with mass more than 600 g from 300 randomly chosen rock melons. [2]

(c) Comment whether the combined masses of the melons of both types is normally distributed. [1]

Rock melons and watermelons are priced at \$3/kg and \$2.80/kg respectively.

(d) Find the probability that the mean selling price of 4 randomly chosen rock melons differs from the selling price of a randomly chosen watermelon by at most 60 cents. [4]

(e) Ah Guan buys 20 melons, n of them are rock melons and the rest are watermelons. Find the greatest value of n such that the probability that the total cost of these 20 melons exceeding \$38 is more than 0.95. You should show your working clearly. [4]

11 Emma has a computer program that generates a random positive integer X. The probability distribution of X is :

$$\mathrm{P}(X=r)=\frac{a}{r^3},\quad r\in\mathbb{Z}^+ \text{ and } a \text{ is positive constant.}$$

For the rest of the question, you may use the following results:

$$\sum_{m=1}^{\infty}\frac{1}{m} \text{ does not exist, } \sum_{m=1}^{\infty}\frac{1}{m^2}=1.6449 \text{ and } \sum_{m=1}^{\infty}\frac{1}{m^3}=1.2021.$$

(a) Find the value of a. [2]

(b) Find E(X) and explain why Var(X) cannot be calculated. [2]

(c) Find $\mathrm{P}\left(X\geq 2 \mid X\leq 15\right)$. [3]

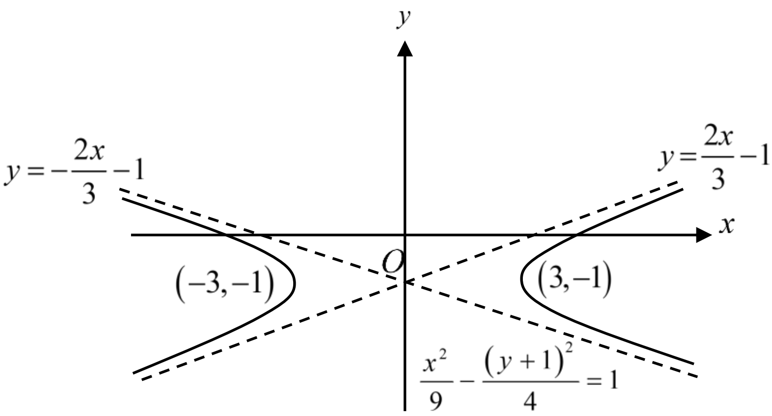
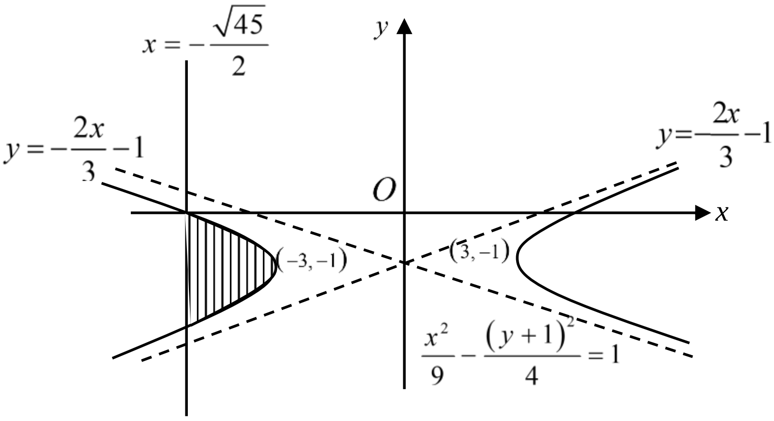
Emma generates 10 numbers using her program namely, $X_1, X_2, \ldots, X_{10}$. The random variable Y denotes the number of times the number ‘3’ occurs among the 10 numbers. You can assume that the numbers generated by the program are independent of each other.

(d) Find $\mathrm{P}(Y>2)$. [2]

(e) Find $\mathrm{P}(X_1+Y=3)$. [4]

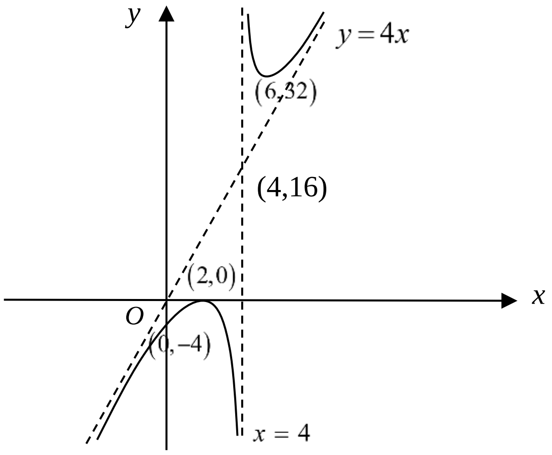
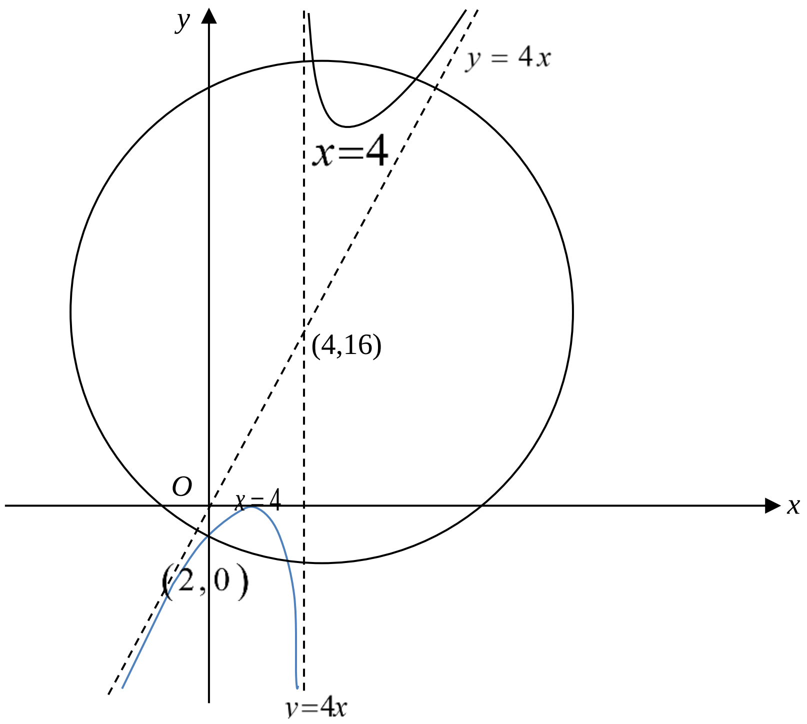
DHS 2022 Year 6 H2 Math Prelim Exam P2 solutions

Section A: Pure Mathematics [40 marks]

Qn	Suggested Solution
1(a)	
(b)	 When $x=-\frac{\sqrt{45}}{2}$ $\frac{(y+1)^2}{4}=\frac{x^2}{9}-1$ $\frac{(y+1)^2}{4}=\frac{9}{36}$ $y+1=\pm 1$ $y=-2 \quad \text{or} \quad y=0$ Required volume $=\pi\left(\frac{\sqrt{45}}{2}\right)^2(2)-9\pi\int_{-2}^{0}\left(1+\frac{(y+1)^2}{4}\right)\mathrm{d}y$ $=9.42 \quad (\text{to 3 s.f.})$

	<u>Alternative (for FM students only)</u> Using shell method, $2\pi\int_{-3}^{-\frac{\sqrt{45}}{2}} x\left(2\sqrt{\frac{4x^2-36}{9}}\right)\mathrm{d}x = 9.42$ (to 3 s.f.)

Qn	Suggested Solution
2(a)	$w^3 = (2-3\mathrm{i})^3$ $= (2)^3 - 3(2)^2(3\mathrm{i}) + 3(2)(3\mathrm{i})^2 - (3\mathrm{i})^3$ $= 8 - 36\mathrm{i} - 54 + 27\mathrm{i}$ $= -46 - 9\mathrm{i}$
(b)	Since w is a root, $(2-3\mathrm{i})^3 - 5(2-3\mathrm{i})^2 + a(2-3\mathrm{i}) + b = 0$ $(-46-9\mathrm{i}) - 5(4-12\mathrm{i}-9) + (2a+b-3a\mathrm{i}) = 0$ $(-21+2a+b) + (51-3a)\mathrm{i} = 0$ $51-3a=0 \Rightarrow a=17$ $-21+2(17)+b=0 \Rightarrow b=-13$
(c)	Since w is a root of $z^3-5z^2+17z-13=0$, $w*$ is also a root as all the coefficients are real. $\therefore z^3-5z^2+17z-13 = (z-w)(z-w^*)(z-\alpha)$ Replace z by $\frac{z}{\mathrm{i}}$, $\left(\frac{z}{\mathrm{i}}\right)^3 - 5\left(\frac{z}{\mathrm{i}}\right)^2 + 17\left(\frac{z}{\mathrm{i}}\right) - 13 = \left(\left(\frac{z}{\mathrm{i}}\right)-w\right)\left(\left(\frac{z}{\mathrm{i}}\right)-w^*\right)\left(\left(\frac{z}{\mathrm{i}}\right)-\alpha\right)$ Multiply by i^3 on both sides, $z^3 - 5\mathrm{i}\,z^2 - 17z + 13\mathrm{i} = (z-\mathrm{i}w)(z-\mathrm{i}w^*)(z-\mathrm{i}\alpha)$ A possible cubic polynomial is $z^3 - 5\mathrm{i}\,z^2 - 17z + 13\mathrm{i}$.

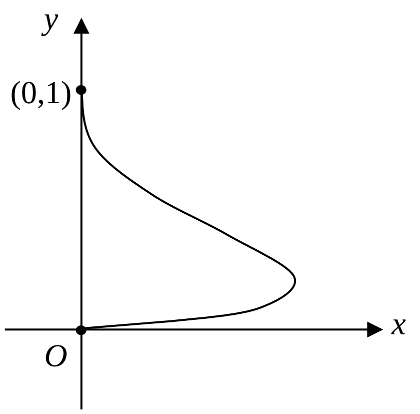
Qn	Suggested Solution
3(a)	
(b)	 $\left(x-4\right)^2+\left(\frac{4\left(x-2\right)^2}{x-4}-16\right)^2=a$ $\left(x-2\right)^2+\left(y-16\right)^2=\left(\sqrt{a}\right)^2$ Therefore, the solution to the equation is the intersection between $y=\frac{4\left(x-2\right)^2}{x-4}$ and circle with radius $\sqrt{a}$ centred at $\left(4,16\right)$ which is the point of intersection between the 2 asymptotes. Distance between $\left(4,16\right)$ and $\left(0,-4\right)$

	$= \sqrt{(4-0)^2 + (16+4)^2} = \sqrt{416}$ Hence, $a > 416$ so that the equation will have 1 negative real root.
(c)	Intersection point of the 2 asymptotes is (4, 16) Thus, $\tan^{-1}(4) < \arg(z - 4 - 16\mathrm{i}) < \frac{\pi}{2}$ Or $-\left(\pi - \tan^{-1}(4)\right) < \arg(z - 4 - 16\mathrm{i}) < -\frac{\pi}{2}$

Qn	Suggested Solutions
4(a)	$\overrightarrow{OA} = \frac{\mu\overrightarrow{OQ} + (1-\mu)\overrightarrow{OP}}{\mu + (1-\mu)}$ $= \mu\begin{pmatrix}0\\2\\-t\end{pmatrix} + (1-\mu)\begin{pmatrix}1\\0\\0\end{pmatrix}$ $= \begin{pmatrix}1-\mu\\2\mu\\-t\mu\end{pmatrix}$ $\overrightarrow{OB} = \frac{\mu\overrightarrow{OR} + (1-\mu)\overrightarrow{OQ}}{\mu + (1-\mu)}$ $= \mu\begin{pmatrix}0\\0\\t\end{pmatrix} + (1-\mu)\begin{pmatrix}0\\2\\-t\end{pmatrix}$ $= \begin{pmatrix}0\\2-2\mu\\-t+2t\mu\end{pmatrix}$ $\overrightarrow{AB} = \begin{pmatrix}0\\2-2\mu\\-t+2t\mu\end{pmatrix} - \begin{pmatrix}1-\mu\\2\mu\\-t\mu\end{pmatrix} = \begin{pmatrix}\mu-1\\2-4\mu\\-t+3t\mu\end{pmatrix}$
(b)	Clearly, $\overrightarrow{OA} \neq k\overrightarrow{OB}$ This means the points are not collinear.

(c)	$\dfrac{\left\lvert \overrightarrow{AB} \bullet \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\left\lVert \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rVert} = \dfrac{1}{5}$ $\dfrac{\left\lvert \begin{pmatrix} \mu - 1 \\ 2 - 4\mu \\ -t + 3t\mu \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rvert}{\left\lVert \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\rVert} = \dfrac{1}{5}$ $3\mu - 3 + 8 - 16\mu = \pm 1$ $13\mu = 4$ or 6 $\mu = \dfrac{4}{13}$ or $\dfrac{6}{13}$
(d)	If angle AOB is a right angle, then $\overrightarrow{OA} \bullet \overrightarrow{OB} = 0$ $\begin{pmatrix} 1 - \mu \\ 2\mu \\ -t\mu \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 2 - 2\mu \\ -t + 2t\mu \end{pmatrix} = 0$ $4\mu - 4\mu^2 + t^2\mu - 2t^2\mu^2 = 0$ **<u>Method 1</u>** $4\mu - 4\mu^2 + t^2\mu - 2t^2\mu^2 = 0$ $\mu = 0$ or $4 - 4\mu + t^2 - 2t^2\mu = 0$ (reject $\because 0 < \mu < 1$) $\mu = \dfrac{4 + t^2}{4 + 2t^2}$ Clearly, $\dfrac{4 + t^2}{4 + 2t^2} > 0$ since $4 + t^2 > 0$ and $4 + 2t^2$ for all $t \in \mathbb{R}$. Since $0 < \mu < 1$, $\dfrac{4 + t^2}{4 + 2t^2} < 1$ $4 + t^2 < 4 + 2t^2$ $t^2 > 0$ Hence $t \in \mathbb{R} \setminus \{0\}$

	Method 2 Since $0<\mu<1$, $4-4\mu+t^2-2t^2\mu=0$ $t^2=\dfrac{4(1-\mu)}{2\mu-1}$ From the graph of t^2 vs μ for $0<\mu<1$, $t^2>0$ or $t^2<-4$ (no solutions for t) then $t\in\mathbb{R}\setminus\{0\}$.

Qn	Suggested Solution
5(a)	$\int \sin x(1-\sin x)\ \mathrm{d}x$ $=\int \sin x-\sin^2 x\ \mathrm{d}x$ $=\int \sin x-\frac{1}{2}(1-\cos 2x)\ \mathrm{d}x$ $=-\cos x-\frac{1}{2}x+\frac{1}{4}\sin 2x+D$ (shown)
(b)	 y-intercepts: $x=\theta(1-\sin\theta)=0$ $\theta=0$ or $\sin\theta=1\Rightarrow\theta=\frac{\pi}{2}$ $\left(\because 0\le t\le\frac{\pi}{2}\right)$ When $\theta=0$, $y=1-\cos 0=0$ When $\theta=\frac{\pi}{2}$, $y=1-\cos\frac{\pi}{2}=1$

(c)

$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=1-\sin\theta-\theta\cos\theta,\ \ \frac{\mathrm{d}y}{\mathrm{d}\theta}=\sin\theta$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin\theta}{1-\sin\theta-\theta\cos\theta}$$

Since tangent at P is parallel to $y=2x$

$$\frac{\sin\theta}{1-\sin\theta-\theta\cos\theta}=2$$

$$3\sin\theta+2\theta\cos\theta-2=0$$

From GC : $\theta=0.42230\ \ \left(\because 0\le t\le \frac{\pi}{2}\right)$

$\therefore P\left(0.249,\ 0.0879\right)$

(d)

Area enclosed

$$=\int_0^1 x\ \mathrm{d}y$$

$$=\int_0^{\frac{\pi}{2}} \theta(1-\sin\theta)\cdot(\sin\theta\ \mathrm{d}\theta)$$

$$=\int_0^{\frac{\pi}{2}} \theta\ \left[\sin\theta(1-\sin\theta)\right]\ \mathrm{d}\theta$$

$$\boxed{\begin{array}{ll} u=\theta, & \dfrac{\mathrm{d}v}{\mathrm{d}\theta}=\sin\theta(1-\sin\theta) \\ \dfrac{\mathrm{d}u}{\mathrm{d}\theta}=1, & v=-\cos\theta-\dfrac{\theta}{2}+\dfrac{\sin 2\theta}{4} \end{array}}$$

$$=\left[\theta\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\ \mathrm{d}\theta$$

$$=\left[\theta\left(-\cos\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)\right]_0^{\frac{\pi}{2}}+\left[\sin\theta+\frac{\theta^2}{4}+\frac{\cos 2\theta}{8}\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\pi}{2}\left(-\frac{\pi}{4}\right)\right]+\left[\left(1+\frac{\pi^2}{16}-\frac{1}{8}\right)-\frac{1}{8}\right]$$

$$=\frac{3}{4}-\frac{\pi^2}{16}$$

(e)

Gradient of $L=2$

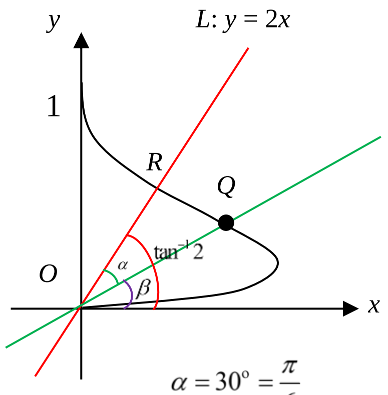
$\Rightarrow$ angle btw L and $x-$axis $=\tan^{-1}2$

$$\beta=\tan^{-1}2-\alpha$$

$$\tan^{-1}\frac{y}{x}=\tan^{-1}2-\frac{\pi}{6}$$

$$\tan^{-1}\frac{1-\cos\theta}{\theta(1-\sin\theta)}=\tan^{-1}2-\frac{\pi}{6}$$

From GC : $\theta=0.596$ rad. (3 sf)



$\alpha=30^{\circ}=\frac{\pi}{6}$

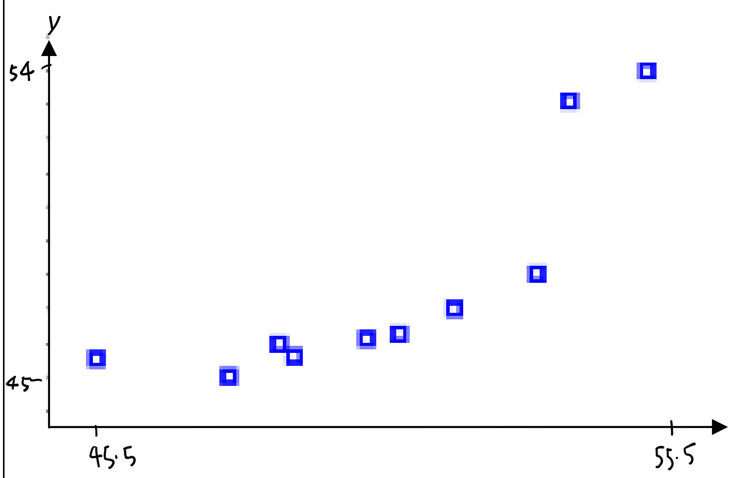
	Alternative Use dot product, $$\cos\frac{\pi}{6}=\frac{\overrightarrow{OR}\bullet\overrightarrow{OQ}}{\left|\overrightarrow{OR}\right|\left|\overrightarrow{OQ}\right|}=\frac{\begin{pmatrix}1\\2\\0\end{pmatrix}\begin{pmatrix}\theta(1-\sin\theta)\\1-\cos\theta\\0\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\right|\left|\begin{pmatrix}\theta(1-\sin\theta)\\1-\cos\theta\\0\end{pmatrix}\right|}=\frac{\theta(1-\sin\theta)+2(1-\cos\theta)}{\sqrt{5}\sqrt{\left(\theta(1-\sin\theta)\right)^2+\left(1-\cos\theta\right)^2}}$$ From GC : $\theta=0.596$ rad. (3 sf) **Note**: The other case where $\tan^{-1}2+\alpha>\frac{\pi}{2}$ need not be considered as there would be no solution.

Section B: Probability and Statistics [60 marks]

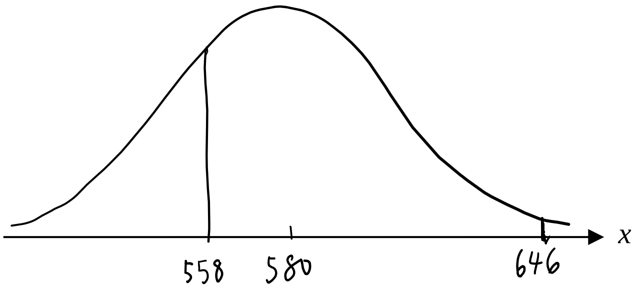
<table>
<tr><th>Qn</th><th>Suggested Solution</th></tr>
<tr><td>6(a)</td><td>Ways = $(7-1)!\times {}^7C_3\times 3!$
= 151200</td></tr>
<tr><td>b(i)</td><td>T Y R A N O S U
R A N
<u>Case 1</u>
All 5 different letters (ie. No identical)
= ${}^8C_5\times 5!$ = 6720

<u>Case 2</u>
2 identical (RR, AA or NN)
= ${}^3C_1\times {}^7C_3\times \frac{5!}{2!}$ = 6300
Total ways = 6720 + 6300 = 13020</td></tr>
<tr><td>b(ii)</td><td><u>Method 1</u>
Reduced sample space = $\frac{1}{{}^8C_2}=\frac{1}{28}$
<u>Method 2a</u>
Conditional probability
$=\frac{\text{P(2R2N} \cap \text{"RAN")}}{\text{P("RAN")}}$
$=\frac{\text{no. of ways (2R2N} \cap \text{"RAN")}}{\text{no. of ways ("RAN")}}$
=
$=\frac{3!}{{}^8C_2\times 3!}$
$=\frac{1}{28}$
<u>Method 2b</u>
Conditional probability
$=\frac{\text{P(2R2N} \cap \text{"RAN")}}{\text{P("RAN")}}$
$=\frac{\frac{2}{11}\times\frac{2}{10}\times\frac{2}{9}\times\frac{1}{8}\times\frac{1}{7}\times 3!}{\frac{2}{11}\times\frac{2}{10}\times\frac{2}{9}\times\frac{1}{8}\times\frac{1}{7}\times {}^8C_2\times 3!}$
$=\frac{1}{28}$</td></tr>
</table>

Qn	Suggested Solution
7(a)(i)	$\mathrm{P}(A \cap B)$ $= \mathrm{P}(\text{fall, rise, rise}) + \mathrm{P}(\text{fall, fall, rise})$ $= (0.4 \times 0.15 \times 0.6) + (0.4 \times 0.85 \times 0.15)$ $= 0.087$
(ii)	$\mathrm{P}(B)$ $= \mathrm{P}(A \cap B) + \mathrm{P}(A' \cap B)$ $= 0.087 + \mathrm{P}(\text{rise, rise, rise}) + \mathrm{P}(\text{rise, fall, rise})$ $= 0.087 + (0.6 \times 0.6 \times 0.6) + (0.6 \times 0.4 \times 0.15)$ $= 0.339$
(iii)	$\mathrm{P}(B \mid A)$ $= \dfrac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)}$ $= \dfrac{0.087}{0.4}$ $= 0.2175$
(b)	Since $\mathrm{P}(B \mid A) = 0.2175 \neq 0.339 = \mathrm{P}(B)$, A and B are not independent.
(c)	Let W be the number of Tuesdays in which the unit price of X rises, out of 12 Tuesdays. $W \sim \mathrm{B}(12, 0.6)$ $\mathrm{P}(W = 5) = 0.101$ (3 s.f.)

Qn	Suggested Solution
8(a)	• Set B will have a larger $\lvert r\rvert$. • The data points for **Set *B* lie relatively closer to a straight line** with negative gradient whereas **Set *A*'s $\lvert r\rvert$ value will be closer to 0** since the data points are more scattered with weak linear correlation between x and y.
(b)(i)	
(ii)	Model C: r = 0.81730 = 0.817 (3 sf) Model D: r = 0.93944 = 0.939 (3 sf) Since $\lvert r\rvert$ value for model D is closer to 1 compared to model C, it indicates a stronger linear correlation. Hence model D is more appropriate.
(iii)	Equation of regression line of y on x for Model D: $y = 45.423 + 8.5357\times10^{-12}\mathrm{e}^{\frac{1}{2}x}$ When $x = 50$, $y = 46.0376$ The mean household expenditure is estimated to be **$46 038** The estimate is reliable since it is an **interpolation** where x = 50 ($45.5 \le x \le 55.5$) and **$\lvert r\rvert$ is close to 1** which indicates a strong positive linear correlation between x and y**.**
(iv)	It is not valid because correlation between income and expenditure does not imply causation.
(v)	Not true, as the product moment correlation coefficient for $y = a + b\mathrm{e}^{\frac{1}{2}x}$ measures the linear correlation between y and $\mathrm{e}^{\frac{1}{2}x}$, not y and $\mathrm{e}^{\frac{1}{20}x}$. **<u>Alternative</u>** Not true. For $y = p + q\mathrm{e}^{\frac{1}{20}x}$**,** the product moment correlation coefficient is 0.8639998 = 0.864 (3.s.f), which is different.

Qn	Suggested Solution
9(a)	Let X be the mass of a randomly chosen mooncake. $H_0 : \mu = 150$ $H_1 : \mu < 150$ where μ is the population mean mass of mooncakes. Since sample size of 9 is small, assume X follows a normal distribution. Under H_0, $\bar{X} \sim N\left(150, \frac{6.73^2}{9}\right)$ From GC, p-value = 0.186322 = 0.186 (3 s.f.) Since the p-value > 0.1, we do not reject H_0 and conclude that there is insufficient evidence at the 10% significance level that the mean mass of the mooncake is less than 150 g, i.e. insufficient evidence to reject owner's claim.
9(b)	Let Y be the working hours of a randomly chosen teacher in the school. $s^2 = \frac{n}{n-1}(\text{sample variance}) = \frac{50k^2}{49}$ hours^2 $H_0 : \mu = 60$ $H_1 : \mu \neq 60$ Under H_0, $\bar{Y} \sim N\left(60, \frac{k^2}{49}\right)$ approximately by Central Limit Theorem since sample size of 50 is large. In order to reject H_0, p-value = $2\,P(\bar{Y} \geq 62) \leq 0.05$ From GC (graph), $0 < k \leq 7.14299$ Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$. **Alternative** In order to reject H_0, $\bar{y}$ must lie within the critical region. i.e, $\bar{y} \geq \bar{y}_{\text{critical}}$ $\therefore \bar{y}_{\text{critical}} \leq 62$ From GC (graph), $0 < k \leq 7.14$ (to 3sf) Set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 7.14\}$.

Qn	Suggested Solution
10(a)	
(b)	$X \sim \mathrm{N}(580, 22^2)$ Expected number $= 300 \times \mathrm{P}(X > 600)$ $= 300 \times 0.18165$ $= 54.495$ $= 54.5$ (3 s.f.)
(c)	No. By combining the masses, it would give a distribution with 2 peaks instead of a single peak.
(d)	Let K and L be the selling price of a randomly chosen rock melon and watermelon respectively. $K = 0.003X, \quad L = 0.0028Y$ $K \sim \mathrm{N}(0.003 \times 580,\ 0.003^2 \times 22^2)$ $K \sim \mathrm{N}(1.74, 0.004356) \Rightarrow \bar{K} \sim \mathrm{N}\left(1.74, \dfrac{0.004356}{4}\right)$ $L \sim \mathrm{N}(0.0028 \times 870,\ 0.0028^2 \times 30^2)$ $L \sim \mathrm{N}(2.436, 0.007056)$ $\bar{K} - L \sim \mathrm{N}(-0.696, 0.008145)$ $\mathrm{P}(\lvert \bar{K} - L \rvert \le 0.60)$ $= \mathrm{P}(-0.60 \le \bar{K} - L \le 0.60)$ $= 0.14373$ $= 0.144$ (3s.f.)
(e)	$K_1 + \ldots + K_n \sim \mathrm{N}(1.74n, 0.004356n)$ $L_1 + \ldots + L_{20-n} \sim \mathrm{N}(2.436(20-n), 0.007056(20-n))$ Let W be the total cost of the 20 melons. $W = K_1 + \ldots + K_n + L_1 + \ldots + L_{20-n}$ $W \sim \mathrm{N}(1.74n + 2.436(20-n), 0.004356n + 0.007056(20-n))$ $\mathrm{P}(W > 38) > 0.95$ Using GC table, $n = 13, \quad \mathrm{P}(W > 38) = 1 > 0.95$ $n = 14, \quad \mathrm{P}(W > 38) = 0.9988 > 0.95$ $n = 15, \quad \mathrm{P}(W > 38) = 0.8113 < 0.95$ Greatest n = 14

Qn	Suggested Solution
11(a)	$\sum_{r=1}^{\infty} \mathrm{P}(X=r)=1$ $\sum_{r=1}^{\infty} \frac{a}{r^3}=1$ $a=\frac{1}{1.2021}=0.83188=0.832$ (3 sf)
(b)	$\mathrm{E}(X)=\sum_{r=1}^{\infty} r\ \mathrm{P}(X=r)=a\sum_{r=1}^{\infty}\frac{1}{r^2}=1.37$ (3 s.f) $\mathrm{E}(X^2)=\sum_{r=1}^{\infty} r^2\ \mathrm{P}(X=r)=a\sum_{r=1}^{\infty}\frac{1}{r}$ does not exist. Therefore Var(X) cannot be calculated.
(c)	Method 1 $\mathrm{P}\left(X\ge 2 \mid X\le 15\right)$ $=1-\mathrm{P}\left(X=1 \mid X\le 15\right)$ $=1-\frac{\mathrm{P}\left(X=1\right)}{\mathrm{P}\left(X\le 15\right)}$ $=1-\frac{a}{\sum_{r=1}^{15}\frac{a}{r^3}}$ $=0.16665=0.167$ (3 s.f) Method 2 $\mathrm{P}\left(X\ge 2 \mid X\le 15\right)$ $=\frac{\mathrm{P}\left(2\le X\le 15\right)}{\mathrm{P}\left(X\le 15\right)}$ $=\frac{\sum_{r=2}^{15}\frac{a}{r^3}}{\sum_{r=1}^{15}\frac{a}{r^3}}$ $=0.16665=0.167$ (3 s.f)
(d)	$Y\sim \mathrm{B}(10,\mathrm{P}(X=3))$ where $\mathrm{P}(X=3)=\frac{a}{27}=\frac{0.83188}{27}=0.030810$ $\mathrm{P}(Y>2)=1-\mathrm{P}(Y\le 2)=0.00298$
(e)	Note: X_1 and Y are dependent variables. **<u>Case 1:</u>** $X_1=1$ and $Y=2$ The first number must be a '1' and the rest of 9 numbers must have two '3's. $\mathrm{P}(X=1)\left[\binom{9}{2}\left(\mathrm{P}(X=3)\right)^2\left(1-\mathrm{P}(X=3)\right)^7\right]=0.022835$ **<u>Case 2:</u>** $X_1=2$ and $Y=1$ The first number must be a '2' and the rest of 9 numbers must have one '3'. $\mathrm{P}(X=2)\left[\binom{9}{1}\left(\mathrm{P}(X=3)\right)\left(1-\mathrm{P}(X=3)\right)^8\right]=0.022448$ **<u>Note:</u>** $X_1=3$ and $Y=0$ This case is impossible as Y is counting the number of '3' generated, probability is 0 for this case. $\therefore \mathrm{P}(X_1+Y=3)=0.022835+0.022448+0=0.0453$ (3 sf)