## Section A: Pure Mathematics [40 marks]

**1** **(i)** The equation $3z^3-7z^2+17z+m=0$, where $m$ is a real constant, has a root $z=1+2\mathrm{i}$. Find the value of $m$.
Hence using an algebraic method, find all the roots of the equation $3z^3-7z^2+17z+m=0$. Show your working clearly. [4]

**(ii)** Hence, solve the equation $\dfrac{3}{w^3}+\dfrac{7}{w^2}+\dfrac{17}{w}-m=0$, giving your answers in the form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. [2]

**2** Relative to the origin $O$, the points $A$, $B$ and $C$ have position vectors **a**, **b** and **c** respectively. It is given that $\lambda$ and $\mu$ are non-zero numbers such that $\lambda\mathbf{a}+\mu\mathbf{b}-\mathbf{c}=\mathbf{0}$ and $\lambda+\mu=1$.

**(i)** Show that the points $A$, $B$ and $C$ are collinear. [3]

The angle between **a** and **b** is known to be obtuse and that $|\mathbf{a}|=2$.

**(ii)** If $k$ denotes the area of triangle $OAB$, show that $(\mathbf{a}\cdot\mathbf{b})^2=4\left(|\mathbf{b}|^2-k^2\right)$. [3]

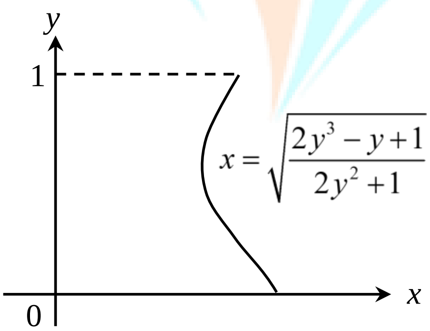
$D$ is a point on the line segment $AB$ with position vector **d**.

**(iii)** It is given that area of triangle $OAB$ is 6 units$^2$, $|\mathbf{b}|=10$ and that $AOD$ is $90^0$. By finding the value of $\mathbf{a}\cdot\mathbf{b}$, find **d** in terms of **a** and **b**. [4]

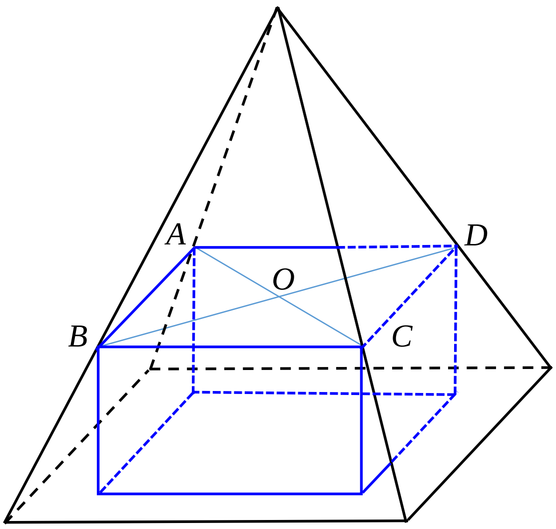
**3** **(a)(i)** Use the substitution $u=1+x^2$ to find $\displaystyle\int\frac{x\mathrm{e}^{1+x^2}}{\sqrt{1+\mathrm{e}^{1+x^2}}}\,\mathrm{d}x$. [4]

**(ii)** Curves $C_1$ and $C_2$ have equations $y=x\mathrm{e}^{x^2-2}-\dfrac{1}{2\mathrm{e}}$ and $y=\dfrac{x\mathrm{e}^{1+x^2}}{\sqrt{1+\mathrm{e}^{1+x^2}}}-\dfrac{1}{2\mathrm{e}^2}$ respectively. The region bounded by the curves $C_1$ and $C_2$, the $y$-axis and the line $x=1$ is $R$. Find the exact area of $R$. [3]

**(b)** The shape of a vase is formed by rotating the part of the curve $x=\sqrt{\dfrac{2y^3-y+1}{2y^2+1}}$ between $y=0$ and $y=1$ through $2\pi$ radians about the $y$-axis (see diagram below). Find the exact volume of the vase formed. [5]



**4**




The product engineer of a factory crafted the design of a rectangular box, using a right pyramid, that is shown on the diagram above (not drawn to scale). The rectangular box is contained in a right pyramid with a rectangular base such that the upper four corners of the box *A*, *B*, *C* and *D* touch the slant faces of the pyramid, and the bottom four corners lie on the base of the pyramid. *O* is the point of intersection of the two diagonals, *AC* and *BD*.

The height of the pyramid is $3\sqrt{2}$ units, the length of the diagonal of its rectangular base is $12\sqrt{2}$ units, the height of the box is $b$ units, where $b < 3\sqrt{2}$, and the angle *AOB* is $\theta$ radians. It is given that the box is made of material with negligible thickness.

**(i)** By finding the length of *OA* in terms of *b*, show that the volume *V* of the rectangular box is given by $V = 8b\left(3\sqrt{2} - b\right)^2 \sin\theta$. [3]

For the rest of the question, it is given that $\theta = \dfrac{\pi}{3}$.

**(ii)** Find the exact value of *b* which maximises *V*. Hence find the cost of manufacturing one such box if the material used to make the box cost \$0.03 per unit$^2$. [6]

When the height of the box is at half the height of the pyramid, it is reducing at a rate of 2 units per second.

**(iii)** Determine whether the volume of the box is expanding or shrinking and find the rate at which this is happening. [3]

## Section B: Probability and Statistics [60 marks]

**5** Two families, each consisting of an adult couple and three children visited a carnival together.

The 10 people went to queue for a ride randomly in one straight line.

**(i)** Find the probability that members of the 2 families stand in alternate positions in that queue. [2]

If the ride is made up of two identical circular carriages of five identical seats each.

**(ii)** Find the number of ways the 10 people can be seated if not all the family members are seated together in the same carriage. [3]

**6** In a soccer practice, the coach instructs the players to practise their penalty kicks. A player scores if he successfully kicks a ball into the net of a goal post. The probability that a player scores on the first kick is $\frac{2}{5}$. For all the subsequent kicks, the probability of scoring on that kick will be $\frac{4}{5}$ if the player scores in the preceding kick, and the probability of scoring on that kick will be $\frac{1}{6}$ if the player did not score in the preceding kick.

**(i)** Owen kicked the ball three times consecutively for his practice. Find the probability that he scored on the third kick, given that he scored only twice out of the three kicks. [3]

**(ii)** Three players each kicked the ball four times consecutively for their practices. Find the probability that one of the players scored on all four kicks, another player scored on the first kick only, while the remaining player only scored on the second and third kicks. [3]

**7** Grade A and grade B sugar produced by a company are packed and sold in packets. The mass of both grade A and grade B sugar sold follows independent normal distributions with mean 2.05 kg. The standard deviation for the mass of a randomly chosen packet of grade A and grade B sugar are 0.025 kg and $\sigma$ kg respectively.

If the probability that the mass of a randomly chosen packet of grade B sugar being less than 2 kg is 0.01,

**(i)** show that the value of $\sigma$ is 0.021493 correct to 5 significant figures. [2]

It is given that the profit per kilogram of grade A and B sugar sold is 50 cents and 40 cents respectively.

**(ii)** Find the probability that the total profit of three randomly chosen packets of grade A sugar is higher than three times the profit of a randomly chosen packet of grade B sugar by not more than 65 cents. [3]

**(iii)** Two packets of grade A sugar and $n$ packets of grade B sugar are selected at random. Find the smallest value of $n$ such that the probability that the mean mass of these packets being less than 2.06 kg is at least 0.97. [3]

**8** In a public swimming centre, the time spent by a randomly chosen user in using its facilities is $T$ minutes, is known to be normally distributed. The centre manager claims that its users spend an average of 50 minutes to use its facilities. To check this claim, time spent by a random sample of 60 users were recorded. The data recorded has an average of 47 minutes and a standard deviation of 16.4 minutes.

**(i)** Find an unbiased estimate of the population variance, giving your answer correct to 2 decimal places. [1]

**(ii)** Test, at the 5% significance level, whether the centre manager overstated the average time spent. [4]

**(iii)** Another sample of size $n$ ($n > 30$) that was collected independently is now used to test, at the 5% significance level, whether the centre manager's claim is valid. For this sample, the mean time taken is 46 minutes. If the result of the test using this information and the unbiased estimate of the population variance in part **(i)** is that the null hypothesis is rejected, find the least possible value that $n$ can take. [4]

**9** **(a)** A random variable $X$ has a binomial distribution with $n = 10$ and probability of success $p$, where $p < 0.5$.

**(i)** Given that $P(X = 3 \text{ or } 4) = 0.2$, write down an equation for the value of $p$, and find this value numerically. [2]

It is given that $p = \frac{1}{5}$.

**(ii)** The mean and standard deviation of $X$ are denoted by $\mu$ and $\sigma$ respectively. Find $P(\mu - \sigma < X < \mu + \sigma)$, correct to 2 decimal places. [3]

**(b)** Mr Chua attempts an online sudoku puzzle each day. The probability that he manages to solve a puzzle on any given day is 0.75, independently of any other day.

**(i)** Find the probability that he solves his third puzzle on the eighth day of his attempt. [2]

**(ii)** Find the probability that, over a period of 8 weeks, Mr Chua manages to solve at least 4 puzzles each week. [2]

**10** A bag contains nine numbered discs. Three discs are numbered 3, four discs are numbered 4 and two discs are numbered −1. Two discs are drawn simultaneously. The sum of numbers on them, denoted by $X$, is recorded.

**(i)** Find the probability distribution for $X$. [3]

**(ii)** Find $E(X)$ and $Var(X)$. [2]

**(iii)** Two independent observations of $X$ are taken. Find the probability that the difference between these two values is at most 5. [3]

**(iv)** Fifty independent observations of $X$ are taken. Find the approximate probability that the sum of these fifty observations is between 250 and 260. [3]

**11** Research is being carried out to study the degradation of a herbicide in soil. The concentration (in percentage) of the herbicide in the soil measured over a period of 120 days is recorded. The observations are listed in the table below. It is given that one of the observations has been recorded wrongly.

| | | | | | | |
|---|---|---|---|---|---|---|
| Number of days ($d$) | 20 | 40 | 60 | 80 | 100 | 120 |
| Concentration ($c$) | 60 | 57 | 41 | 36 | 33 | 31 |

**(i)** Draw a scatter diagram to illustrate the data and circle the incorrect observation. [3]

For the rest of the question, you should exclude the incorrect observation.

**(ii)** Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. [2]

It is thought that this set of data can be modelled by one of the following formulae after removing the incorrect observation.

Model A: $c^2 = a + bd$

Model B: $c = ae^{bd}$

**(iii)** By calculating the product moment correlation coefficients, explain clearly which of the above models is a more appropriate model for this set of data. [3]

**(iv)** Use the model you identified in **(iii)** to find the equation of a suitable regression line and use your equation to estimate the concentration of the herbicide in the soil after 140 days. [2]

**(v)** Comment on the reliability of the estimate obtained in **(iv)**. [1]

**(vi)** Give an interpretation of the vertical intercept of the regression line obtained in **(iv)** in the context of the question. [1]

**End of Paper**

**1** **(i)** For positive real constant $c$, state a sequence of three transformations in terms of $c$, that will transform the graph with equation of the form $y = \mathrm{f}(2x+3)+c$ onto the graph with equation $y = \mathrm{f}(x)$. [3]

**(ii)** The point with coordinates $(-2, 0)$ that lies on the curve with equation of the form $y = \mathrm{f}(2x+3)+c$ is mapped onto the point with coordinates $(0, -1)$ that is on the curve with equation $y = \mathrm{f}(x)$. State the value of $c$. [1]

**2** The complex numbers $z_1$, $z_2$ and $z_3$ are given by $z_1 = \left(1-\sqrt{3}\mathrm{i}\right)^2$,

$$z_2 = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)\right]^6 \text{ and } z_3 = -1+\sqrt{3}\mathrm{i}.$$

**(i)** Using an algebraic method, find $\dfrac{z_2}{z_1}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $\theta$ is an exact real constant such that $-\pi < \theta \le \pi$. [3]

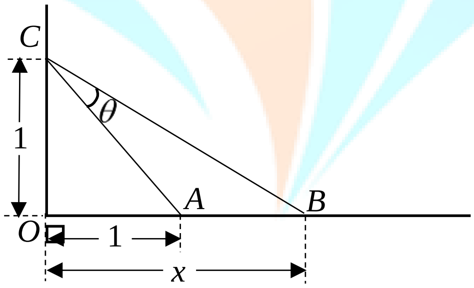
**(ii)** Hence find $\dfrac{z_2}{z_1} + z_3$ in the form $p\mathrm{e}^{\mathrm{i}\alpha}$, where both $r$ and $\theta$ are exact real constants such that $r > 0$ and $-\pi < \theta \le \pi$. [3]

**3** It is given that the curve $C$ has equation $y = \dfrac{x^2 - x + 7}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$.

**(i)** Without using a calculator, find the set of values that $y$ cannot take. [3]

**(ii)** Sketch $C$, stating clearly the equations of any asymptotes, the coordinates of the stationary points and the point(s) where the curve crosses the axes. [3]

**4** **(i)** Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta + \frac{1}{3}\theta^3$. You may use the standard results given in the List of Formulae (MF26). [2]




In the right-angle triangle $OBC$ shown above, point $A$ lies on $OB$ such that $OA = 1$, $OB = x$, where $x > 1$ and $OC = 1$. It is given that angle $COB$ is $\dfrac{\pi}{2}$ radians and that angle $ACB$ is $\theta$ radians (see diagram).

**(ii)** Show that $AB = \dfrac{2\tan\theta}{1-\tan\theta}$. [2]

**(iii)** Given that $\theta$ is a sufficently small angle, show that

$$AB \approx a\theta + b\theta^2 + c\theta^3$$

for exact real constants $a$, $b$ and $c$ to be determined. [3]

**5** **(i)** By considering $u_n - u_{n+1}$, where $u_n = \dfrac{1}{n(n+1)(n+2)}$,

find $\displaystyle\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)}$ in terms of $N$. [3]

**(ii)** Hence or otherwise, find $\displaystyle\sum_{n=5}^{N+3} \frac{1}{n(n-1)(n-2)(n-3)}$. [3]

**(iii)** Deduce that

$$\frac{1}{6^2} + \frac{1}{12^2} + \frac{1}{20^2} + \frac{1}{30^2} + \frac{1}{42^2} + \ldots$$

is less than $\dfrac{1}{18}$. Show your workings clearly. [3]

**6** **(a)** Find $\displaystyle\int \frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x$ for $0 < x < 1$. [3]

**(b)** **(i)** Sketch the graphs of $y = \left|x^2 - 7\right|$ and $y = x + 5$ on the same diagram. Indicate clearly the $x$-intercepts and the values of $x$ where the two curves intersect. Hence solve the inequality $\left|x^2 - 7\right| \geq x + 5$. [4]

**(ii)** Hence, for $a > 5$, find $\displaystyle\int_3^a \left|\,\left|x^2 - 7\right| - x - 5\right| \mathrm{d}x$ in terms of $a$. Leave your answer in exact form. [3]

**7** A curve $C$ has parametric equations

$$x = \sin^3 t\,, \quad y = \cos^2 t, \quad -\frac{\pi}{2} < t < 0\,.$$

The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\dfrac{\pi}{2} < p < 0$, meets the $x$-axis and $y$-axis at $Q$ and $R$ respectively.

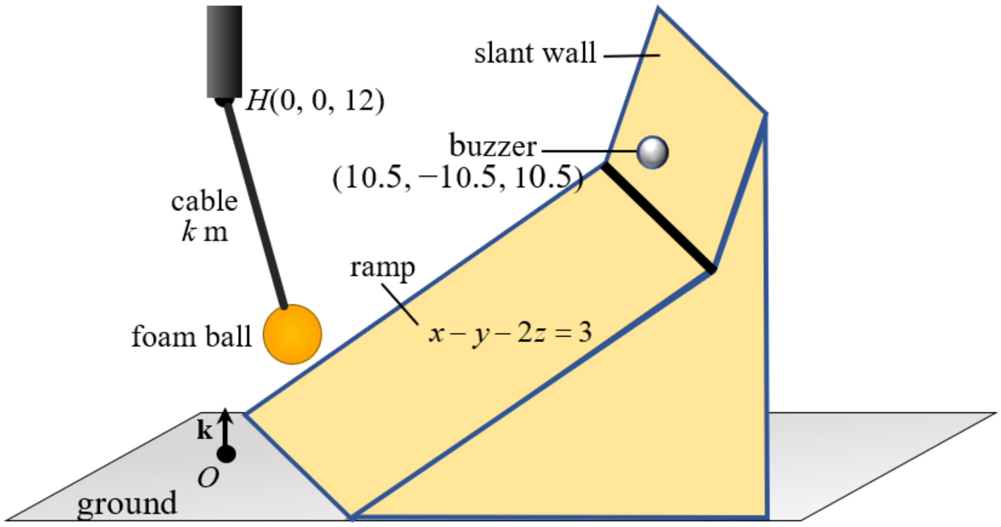
**(i)** By finding the equation of the tangent at the point $P$, show that the area of the triangle $OQR$ is $-\dfrac{1}{12}\sin p\,(2+\cos^2 p)^2$. [6]

**(ii)** Find a cartesian equation of the locus of the mid-point of $QR$ as $p$ varies. You need not indicate its domain. [5]

| | | | |
|---|---|---|---|
| **8** | **(a)** Functions f and g are defined by<br>$\mathrm{f}: x \mapsto x^2, \quad x < 0,$<br>$\mathrm{g}: x \mapsto \dfrac{1}{x}, \quad x > 0.$ | | |
| | **(i)** Explain why the composite function gf exists. | [1] | |
| | **(ii)** Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly. | [3] | |
| | **(b)** For real values *a*, the function h is defined by<br>$\mathrm{h}: x \mapsto ax - \dfrac{1}{x}, \quad x < 0.$ | | |
| | **(i)** If $a$ is negative, explain clearly with a well-labelled diagram, why $\mathrm{h}^{-1}$ does not exist. | [4] | |
| | **(ii)** If $a = 1$, find $\mathrm{h}^{-1}$ in similar form. | [3] | |

| | | |
|---|---|---|
| **9** | **(a)** An arithmetic progression has first term $a$ and common difference $d$, where $a > 0$ and $d \neq 0$. The eighth, third and second term of the progression are the first three terms of an infinite geometric progression. | |
| | **(i)** Find the common ratio of the geometric progression. | [3] |
| | **(ii)** Find the exact sum of the odd-numbered terms of the geometric progression in terms of $a$. | [3] |
| | **(b)** A programmer coded a program involving a rabbit-fox chase along a straight path to model the actual hunt for a rabbit by a fox.<br>The rabbit first hop is 1.75 m. In each subsequent hop, the distance covered is 1% less than its previous hop. The fox first leaps 3 m. In each subsequent leap, the distance covered is 0.02 m less than its previous leap. Initially the rabbit is 60 m ahead of the fox and assume that the rabbit and the fox start and end each hop and leap at the same time. | |
| | **(i)** By finding the total distance travelled by the fox and the rabbit after $n$ leaps and hops respectively, find the minimum number of hops and leaps for the fox to catch up with the rabbit. | [4] |
| | **(ii)** Find the number of leaps the fox takes before it comes to a stop. Hence, find the minimum starting distance, in metre, between the fox and the rabbit such that the fox will never catch up with the rabbit. Leave your answer to the nearest integer. | [2] |

**10** The production team of a popular variety show, *Sprinting Man*, is preparing a site for a segment of the show. In this segment, each participant is to sprint from the starting point, go up a ramp and press a buzzer to complete the challenge.




Referring the starting point as the origin $O$ and the horizontal ground as the $x$-$y$ plane, the top surface of the ramp has equation $x - y - 2z = 3$ (see diagram that is not drawn to scale). Distances are measured in metres.

**(i)** Find the angle of inclination of the ramp. [2]

A spherical polyurethane foam ball of radius 1 m is suspended from a point $H$ with coordinates $(0, 0, 12)$ by a cable of length $k$ m, that is taut all the time. The ball will be swung in various directions during the challenge to increase the level of difficulty.

**(ii)** If the production team wants to ensure that the foam ball will never come in contact with the ramp, find the range of values that $k$ can take. [3]

The buzzer that the participants are to press is located at the point with coordinates $(10.5, -10.5, 10.5)$. This point lies on a flat slant wall which intersects the ramp along the line $l$ with cartesian equation $x = y + 20,\ z = 8.5$.

**(iii)** Find a cartesian equation of the slant wall. [3]

A camera is to be placed along a line $L$ with equation $\mathbf{r} = 12\mathbf{k} + t(\mathbf{i} + 3\mathbf{j}),\ t \in \mathbb{R}$, with its position denoted by $C$.

**(iv)** If the camera is at a distance of $\sqrt{254}$ m from a point $P$ with coordinates $(10, -10, 10)$, determine the possible coordinates of $C$ exactly, showing your workings. Hence deduce the point on $L$ that is nearest to $P$. [4]

**11** The cylindrical tank in a research laboratory has a cross-sectional area of $4\text{ m}^2$. To cool the tank, water is pumped in and out of the tank simultaneously. The volume and height of the water in the tank at any time $t$ minutes is given by $V$ (litres) and $h$ (metres) respectively. Clean water is pumped into the tank at a rate that is proportional to $h^2$ and the water is pumped out from the tank at a rate that is proportional to $h$.

**(i)** Assume that the water does not overflow and that there is no change to the height of the water when $h$ is 10, show that $\frac{dh}{dt} = \frac{kh(h-10)}{4}$ where $k$ is a real constant. [4]

The tank was initially filled with clean water to a height of 2 metres. When the height of the water is 5 metres, the volume of water is increasing at a rate of 5.5 litres per minute.

**(ii)** Find the exact value of $k$. Hence find $h$ in terms of $t$. [5]

**(iii)** Sketch a graph of $h$ against $t$. Hence write down the minimum height of the cylindrical tank that will not result in the overflow of the water. [3]

# ANDERSON SERANGOON JUNIOR COLLEGE

**MATHEMATICS** **9758**

**H2 Math Prelim Paper 2 (100 marks)**

**19 Sept 2022**

**3 hours**

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

**READ THESE INSTRUCTIONS FIRST**

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

| Question number | Marks |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| | |
| Total | |

This document consists of 22 printed pages and 2 blank pages.

**[Turn Over**

| | Section A: Pure Mathematics [40 marks] | |
|---|---|---|
| 1 | **(i)** The equation $3z^3-7z^2+17z+m=0$, where $m$ is a real constant, has a root $z=1+2\mathrm{i}$. Find the value of $m$. Hence using an algebraic method, find all the roots of the equation $3z^3-7z^2+17z+m=0$. Show your working clearly. | [4] |
| | **(ii)** Hence, solve the equation $\frac{3}{w^3}+\frac{7}{w^2}+\frac{17}{w}-m=0$, giving your answers in the form $a+b\mathrm{i}$, where $a,b\in\mathbb{R}$. | [2] |
| | | |
| | **Solution** | |
| | (i) Since $z=1+2\mathrm{i}$ is a root, | |
| | $3(1+2\mathrm{i})^3-7(1+2\mathrm{i})^2+17(1+2\mathrm{i})+m=0$ | |
| | $3(-11-2\mathrm{i})-7(-3+4\mathrm{i})+17+34\mathrm{i}+m=0$ | |
| | $5+m=0$ | |
| | $m=-5$ | |
| | $3z^3-7z^2+17z-5=0$ | |
| | Since coefficients are all real, $z=1-2\mathrm{i}$ is also a root. | |
| | $(z-(1+2\mathrm{i}))(z-(1-2\mathrm{i}))(3z-k)=3z^3-7z^2+17z+m$ | |
| | $(z^2-2z+5)(3z-k)=3z^3-7z^2+17z+m$ | |
| | Comparing, $5(-k)=-5$ | |
| | $k=1$ | |
| | $(z-(1+2\mathrm{i}))(z-(1-2\mathrm{i}))(3z-1)=0$ | |
| | $\therefore z=1+2\mathrm{i},\ 1-2\mathrm{i},\ \frac{1}{3}$ | |
| | Alternatively, | |
| | Since coefficients are all real, so $z=1-2\mathrm{i}$ is also a root. | |
| | $\Rightarrow(z-(1+2\mathrm{i}))(z-(1-2\mathrm{i}))(3z-k)=3z^3-7z^2+17z+m=0$ | |
| | $(z^2-2z+5)(3z-k)=3z^3-7z^2+17z+m$ | |
| | Comparing coefficients of $z$ | |
| | $-2(-k)+15=17$ | |
| | $k=1$ | |
| | $(z^2-2z+5)(3z-1)=0$ | |
| | Comparing, $m=5(-1)=-5$ | |
| | $\therefore z=1+2\mathrm{i},\ 1-2\mathrm{i},\ \frac{1}{3}$ | |
| | **(ii)** $\frac{3}{w^3}+\frac{7}{w^2}+\frac{17}{w}+5=0$ | |
| | $-\frac{3}{w^3}-\frac{7}{w^2}-\frac{17}{w}-5=0$ | |
| | $\frac{3}{(-w)^3}-\frac{7}{(-w)^2}+\frac{17}{(-w)}-5=0$ | |

**Commented [KW(W1]: Method**
Students should not use the GC to find the roots as the question requires an algebraic method.

**Commented [KW(W2]: Method**
Students are required to use the answers in (i) to solve (ii).

**Commented [KW(W3]: Concepts**
Some students did not recognize that the coefficient of $z^3$ is 3 and wrote $z$-$k$ instead.

Some others did not know how to express the cubic expression as a product of linear factors.

The highest power is 3 so there should only be 3 roots.

$3\left(-\frac{1}{w}\right)^3-7\left(-\frac{1}{w}\right)^2+17\left(-\frac{1}{w}\right)-5=0$

Let $z=-\frac{1}{w}$

From (i), $-\frac{1}{w}=1+2\mathrm{i}$ or $-\frac{1}{w}=1-2\mathrm{i}$ or $-\frac{1}{w}=\frac{1}{3}$

$w=-\frac{1}{1+2\mathrm{i}}$ or $w=-\frac{1}{1-2\mathrm{i}}$ or $w=-3$

$\therefore w=-\frac{1}{5}(1-2\mathrm{i}),\ -\frac{1}{5}(1+2\mathrm{i}),\ -3$

**Commented [KW(W4]: Concept**
Note that the sign for the first and third term should be positive so substitution should be -1/w instead of 1/w.

**Commented [KW(W5]: Concept**
Some students did not know how to simplify $-\frac{1}{1+2\mathrm{i}}$ and $-\frac{1}{1-2\mathrm{i}}$.

2 Relative to the origin $O$, the points $A$, $B$ and $C$ have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\lambda$ and $\mu$ are non-zero numbers such that $\lambda\mathbf{a}+\mu\mathbf{b}-\mathbf{c}=0$ and $\lambda+\mu=1$.

(i) Show that the points $A$, $B$ and $C$ are collinear. [3]

The angle between $\mathbf{a}$ and $\mathbf{b}$ is known to be obtuse and that $|\mathbf{a}|=2$.

(ii) If $k$ denotes the area of triangle $OAB$, show that $(\mathbf{a}\cdot\mathbf{b})^2=4\left(|\mathbf{b}|^2-k^2\right)$. [3]

$D$ is a point on the line segment $AB$ with position vector $\mathbf{d}$.

(iii) It is given that area of triangle $OAB$ is 6 units², $|\mathbf{b}|=10$ and that $AOD$ is $90^0$. By finding the value of $\mathbf{a}\cdot\mathbf{b}$, find $\mathbf{d}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

**Commented [LT6]: Misconception**
Some confused it with the concept of coplanar. A, B and C are **collinear** means 3 points are on the **same straight line** while 3 points being **coplanar** means they are on the **same plane**.

It is important to note that Ratio Theorem **is not a method to prove that 3 points are collinear.** It is a result that works on the basis that the 3 points must be on a line before having the position vector of the 3rd point to be expressed in the form taught in the lecture notes.

**Commented [LT7]: Question Reading**
Some used the angle OAB or ABO when it should be AOB. Note that the requirement for dot product is to have the vectors to be converging or diverging.

**Commented [LT8]: Misconception**
Many students did not realize the implication of having the angle to be obtuse means that $\mathbf{a}\cdot\mathbf{b}<0$

**Commented [LT9]: Question Reading**
Some drew the wrong diagram where it was interpreted as $ADO$ is $90^0$

**Solution**

**(i)**

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$= \lambda\mathbf{a} + \mu\mathbf{b} - \mathbf{a}$

$= (\lambda - 1)\mathbf{a} + \mu\mathbf{b}$

$= -\mu\mathbf{a} + \mu\mathbf{b}$

$= \mu(\mathbf{b} - \mathbf{a})$

Since $\overrightarrow{AC} = \mu\,\overrightarrow{AB}$ for some $\mu \in \mathbb{R}$, and $A$ is a common point, therefore $A$, $B$, $C$ are collinear.

**Commented [LT10]:** Presentation of Answer
Many did not make mention of the common point between the two vectors used.

**(ii)** $k = \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$

**Commented [LT11]:** Misconception
Incorrect to write it as $k = \frac{1}{2}(\mathbf{a} \times \mathbf{b})$ or $k = \frac{1}{2}\mathbf{ab}$ as both $\mathbf{a}$ and $\mathbf{b}$ are vectors.

$k = \frac{1}{2}|\mathbf{a}||\mathbf{b}||\sin\theta|$, where $\theta$ is the obtuse angle between $\mathbf{a}$ and $\mathbf{b}$

$k^2 = |\mathbf{b}|^2 \sin^2\theta$

$k^2 = |\mathbf{b}|^2 (1 - \cos^2\theta)$

$k^2 = |\mathbf{b}|^2 \left[1 - \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right)^2\right]$

$k^2 = |\mathbf{b}|^2 - \frac{(\mathbf{a}\cdot\mathbf{b})^2}{4}$

$(\mathbf{a}\cdot\mathbf{b})^2 = 4\left(|\mathbf{b}|^2 - k^2\right)$

**Commented [LT12]:** Misconception
Wrote $(\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2 - 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2$ when it should be $(\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2\cos^2(\angle AOB)$.

**(iii)** Since $D$ lies on line $AB$,

$\mathbf{d} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ for some $\lambda \in \mathbb{R}$

**Commented [LT13]:** Question Reading
Many did not use the fact that point D lines on the line AB.

$OD$ is perpendicular to $OA$

$\Rightarrow [\mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})]\cdot\mathbf{a} = 0$ for some $\lambda \in \mathbb{R}$

$\Rightarrow (1-\lambda)|\mathbf{a}|^2 + \lambda(\mathbf{b}\cdot\mathbf{a}) = 0$

$4(1-\lambda) + \lambda(\mathbf{b}\cdot\mathbf{a}) = 0$

As $(\mathbf{a}\cdot\mathbf{b})^2 = 4\left(|\mathbf{b}|^2 - k^2\right)$

$(\mathbf{a}\cdot\mathbf{b})^2 = 4(10^2 - 6^2)$

$\mathbf{a}\cdot\mathbf{b} = -16$ ($\because \theta$ is obtuse)

**Commented [LT14]:** Misconception
Some thought that dot product must always gi rise to a positive value which is incorrect. The correct definition should be

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \Leftrightarrow \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

And not $\cos\theta = \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|$. We only use

$\cos\theta = \left|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}\right|$ if we are told that the angle is acute or when we are trying to find acute ang

$\Rightarrow 4(1-\lambda) - 16\lambda = 0$

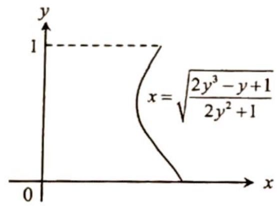
$\lambda = \frac{1}{5}$

$\mathbf{d} = \frac{4}{5}\mathbf{a} + \frac{1}{5}\mathbf{b}$

**3** (a)(i) Use the substitution $u = 1 + x^2$ to find $\int \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}\,dx$. [4]

(ii) Curves $C_1$ and $C_2$ have equations $y = xe^{x^2-2} - \frac{1}{2e}$ and $y = \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}} - \frac{1}{2e^2}$ respectively. The region bounded by the curves $C_1$ and $C_2$, the $y$-axis and the line $x = 1$ is $R$. Find the exact area of $R$. [3]

(b) The shape of a vase is formed by rotating the part of the curve $x = \sqrt{\frac{2y^3 - y + 1}{2y^2 + 1}}$ between $y = 0$ and $y = 1$ through $2\pi$ radians about the $y$-axis (see diagram below). Find the exact volume of the vase formed. [5]




**Solution**

**(ai)** $\frac{du}{dx} = 2x$

$$\int \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}\,dx = \frac{1}{2}\int \frac{e^{1+x^2}}{\sqrt{1+e^{1+x^2}}}(2x)\,dx$$

$$= \frac{1}{2}\int \frac{e^u}{\sqrt{1+e^u}}\,du$$

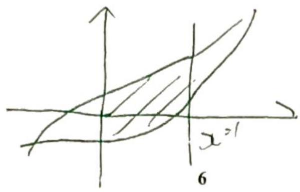
$$= \frac{1}{2}\frac{\left(1+e^u\right)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \left(1+e^{1+x^2}\right)^{\frac{1}{2}} + c$$

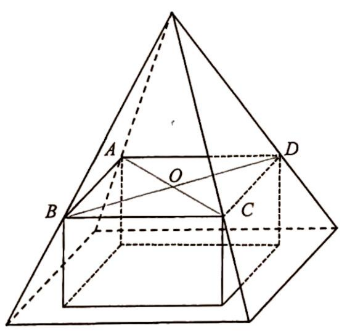
**Commented [TCK15]: Presentation**
There is no need to replace $x$ by $\pm\sqrt{u-1}$.
Simply replace $2x\,dx$ by $du$.
Alternatively,

$$\int \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}\,dx = \int \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}\frac{dx}{du}\,du$$
$$= \int \frac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}\frac{1}{2x}\,du$$
$$= \int \frac{e^u}{\sqrt{1+e^u}}\frac{1}{2}\,du$$

Do not forget to include constant of integration.






| | | |
|---|---|---|
| | (ii) Area $=\int_0^1\left(\dfrac{xe^{1+x^2}}{\sqrt{1+e^{1+x^2}}}-\dfrac{1}{2e^2}\right)dx-\int_0^1\left(xe^{x^2-2}-\dfrac{1}{2e}\right)dx$ | |
| | $=\left[\left(1+e^{1+x^2}\right)^{\frac{1}{2}}\right]_0^1-\left[\dfrac{e^{x^2-2}}{2}\right]_0^1+\left(\dfrac{1}{2e}-\dfrac{1}{2e^2}\right)$ | |
| | $=\sqrt{1+e^2}-\sqrt{1+e}$ | |
| | (b) Volume $=\pi\int_0^1\left(\dfrac{2y^3-y+1}{2y^2+1}\right)dy$ | |
| | $=\pi\int_0^1 y+\dfrac{1-2y}{2y^2+1}\,dy$ | |
| | $=\pi\int_0^1\left(y-\dfrac{2y}{2y^2+1}+\dfrac{1}{2y^2+1}\right)dy$ | |
| | $=\pi\left[\dfrac{y^2}{2}-\dfrac{1}{2}\ln\left(1+2y^2\right)+\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}y\right)\right]_0^1$ | |
| | $=\pi\left(\dfrac{1}{2}-\dfrac{1}{2}\ln 3+\dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}\right)$ | |
| | | |
| 4 |  The product engineer of a factory crafted the design of a rectangular box, using a right pyramid, that is shown on the diagram above (not drawn to scale). The rectangular box is contained in a right pyramid with a rectangular base such that the upper four corners of the box *A*, *B*, *C* and *D* touch the slant faces of the pyramid, and the bottom four corners lie on the base of the pyramid. *O* is the point of intersection of the two diagonals, *AC* and *BD*.<br><br>The height of the pyramid is $3\sqrt{2}$ units, the length of the diagonal of its rectangular base is $12\sqrt{2}$ units, the height of the box is $b$ units, where $b<3\sqrt{2}$, and the angle $AOB$ is $\theta$ radians. It is given that the box is made of material with negligible thickness. | |

**Commented [TCK16]: Concept**
Area of region is bounded by upper curve $C_2$ and lower curve $C_1$ from $x=0$ to $x=1$. Hence the method

$\int_0^1(C_2-C_1)\,dx=\int_0^1 C_2\,dx-\int_0^1 C_1\,dx$

Many are not familiar with the technique to integrate $xe^{x^2-2}$.

**Careless mistake**
Many took $\dfrac{1}{2e^2}$ as $\dfrac{1}{2e}$.

**Commented [TCK17]: Misconception**
Volume is not $2\pi\int_0^1 x^2dy$ or $\pi\int_0^1 x\,dy$ or $\int_0^1 x^2\,dy$ (many left out $\pi$).

**Careless mistakes**
Many did not reduce $\dfrac{2y^3-y+1}{2y^2+1}$ to partial fractions correctly.

Many applied the technique in MF26 wrongly as seen below:

$\int\dfrac{1}{2y^2+1}dy$

$=\int\dfrac{1}{\left(\sqrt{2}y\right)^2+1}dy$

$=\dfrac{1}{1}\tan^{-1}\left(\dfrac{\sqrt{2}y}{1}\right)$ (wrong)

The correct way is

$\int\dfrac{1}{2y^2+1}dy=\dfrac{1}{2}\int\dfrac{1}{y^2+\frac{1}{2}}dy$

$=\dfrac{1}{2}\left(\dfrac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\dfrac{y}{\frac{1}{\sqrt{2}}}\right)=\dfrac{1}{\sqrt{2}}\tan^{-1}\sqrt{2}y$

(i) By finding the length of $OA$ in terms of $b$, show that the volume $V$ of the rectangular box is given by $V = 8b\left(3\sqrt{2}-b\right)^2 \sin\theta$. [3]

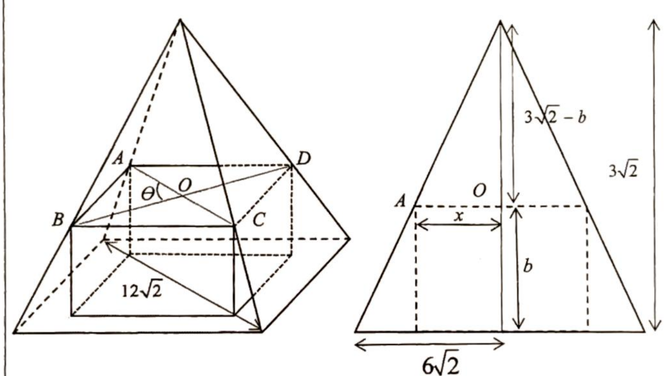
For the rest of the question, it is given that $\theta = \frac{\pi}{3}$.

(ii) Find the exact value of $b$ which maximises $V$. Hence find the cost of manufacturing one such box if the material used to make the box cost \$0.03 per unit$^2$. [6]

When the height of the box is at half the height of the pyramid, it is reducing at a rate of 2 units per second.

(iii) Determine whether the volume of the box is expanding or shrinking and find the rate at which this is happening. [3]

**Solution**




(i) Let $OA = x$ and $V =$ volume of box

By similar triangles, $\frac{x}{2(3\sqrt{2})} = \frac{3\sqrt{2}-b}{3\sqrt{2}} \Rightarrow x = 2\left(3\sqrt{2}-b\right)$

$V = AB \times BC \times b$

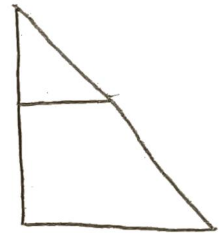
$= \left(2x\sin\frac{\theta}{2}\right)\left(2x\cos\frac{\theta}{2}\right)b$

$= 2b\left[2\left(3\sqrt{2}-b\right)\right]^2 \sin\theta$

$V = 8b\left(3\sqrt{2}-b\right)^2 \sin\theta$ (shown)

(ii) $V = 4\sqrt{3}b\left(3\sqrt{2}-b\right)^2$

$\frac{dV}{db} = 4\sqrt{3}\left[-2b\left(3\sqrt{2}-b\right)+\left(3\sqrt{2}-b\right)^2\right]$



① similar Δs
② TOA CAH SOH
③ pythagolas thm

**Commented [ABK18]: Appproach**
For the volume of the box stated, quite a number of students use the following: $V = 4 \times$ Area of $\Delta AOB \times b$. Since this is a **'show' question**, a detailed explaination is needed why they formula is used. Anything based on assumption (even when the result is obtained) is not accepted.

**Commented [ABK19]: Strategy**
In differentiating $V$ with respect to $b$, some students expanded the RHS into a cubic equation. This is **not efficient** because when solving dV/db=0, we have to factorise again. Please think about the time constraint of the 3 hour paper.

| | | |
|---|---|---|
| | $\frac{dV}{db}=4\sqrt{3}\left(3\sqrt{2}-b\right)\left(3\sqrt{2}-3b\right)$ | |
| | For stationary point, $4\sqrt{3}\left(3\sqrt{2}-b\right)\left(3\sqrt{2}-3b\right)=0$ | |
| | $\Rightarrow b=\sqrt{2}$ or $b=3\sqrt{2}$ (rejected since $b<h$) | |
| | $\frac{d^2V}{db^2}=4\sqrt{3}\left[-2b(-1)+\left(3\sqrt{2}-b\right)(-2)+2\left(3\sqrt{2}-b\right)(-1)\right]$ $=4\sqrt{3}\left[6b-12\sqrt{2}\right]$ $=24\sqrt{3}\left(b-2\sqrt{2}\right)$ | |
| | $\left.\frac{d^2V}{db^2}\right\rvert_{b=\sqrt{2}}=-24\sqrt{6}<0$ | |
| | Thus $V$ is maximised when $b=\sqrt{2}$. | |
| | $BC=4\left(3\sqrt{2}-\sqrt{2}\right)\cos\frac{\pi}{6}=4\sqrt{6}$ $AB=4\left(3\sqrt{2}-\sqrt{2}\right)\sin\frac{\pi}{6}=4\sqrt{2}$ | |
| | Cost $=0.03\times 2\left[4\sqrt{6}\left(4\sqrt{2}\right)+\sqrt{2}\left(4\sqrt{6}\right)+\sqrt{2}\left(4\sqrt{2}\right)\right]$ | |
| | = \$4.64 | |
| | **(iii)** $\frac{dV}{dt}=\frac{dV}{db}\times\frac{db}{dt}$ | |
| | When $b=\frac{3}{2}\sqrt{2}$, | |
| | $\left.\frac{dV}{dt}\right\rvert_{b=\frac{3}{2}\sqrt{2}}=4\sqrt{3}\left(3\sqrt{2}-\frac{3}{2}\sqrt{2}\right)\left(3\sqrt{2}-\frac{9}{2}\sqrt{2}\right)\times(-2\text{ units/s})$ | |
| | $=36\sqrt{3}$ units$^3$/s | |
| | Since $\left.\frac{dV}{dt}\right\rvert_{b=\frac{3}{2}\sqrt{2}}>0$, the volume of the box is expanding. | |
| | | |
| | **Section B: Probability and Statistics [60 marks]** | |
| 5 | Two families, each consisting of an adult couple and three children visited a carnival together. | |
| | The 10 people went to queue for a ride randomly in one straight line. | |
| | **(i)** Find the probability that members of the 2 families stand in alternate positions in that queue. | [2] |
| | If the ride is made up of two identical circular carriages of five identical seats each. | |
| | **(ii)** Find the number of ways the 10 people can be seated if not all the family members are seated together in the same carriage. | [3] |
| | | |
| | **Solution** | |

**Commented [ABK20]:** Misconception
When solving dV/db=0, **there should be 2 values for b** and one of it will be rejected due to the condition given in the question. Some students in the process of solving and factorizing will cancel the factor $\left(3\sqrt{2}-b\right)$. This should not be done. It **should still be considered for part of the solutions** obtained. When needed, it will then be required to be rejected properly.

**Commented [ABK21]:** Inadequate steps
When proving whether $b=\sqrt{2}$ gives the max/min volume, we can use (1) 2nd derivative test (2) 1st derivative test (sign test). Notice that the **value of the 2nd derivative test need to be quoted** as part of the answer.

Students who did using the 1st derivative test (sign test), a number failed to quote the values. **Values must be quoted** to indicate that the slope is either +ve or -ve.

**Commented [ABK22]:** Question Reading/ Interpretation
The question asks for the cost of material used to maximise the volume. **Material used is dependent on the SURFACE AREA** of material and NOT the volume. A number of students found the volume and use this to calculate the cost.

| | | |
|---|---|---|
| | (i) Required probability $= \boxed{\frac{2\times 5!\times 5!}{10!}}$ or $\frac{2\times 5^2\times 4^2\times 3^2\times 2^2\times 1^2}{10!}$ | |
| | $= \frac{1}{126}$ | |
| | | |
| | **(ii)** Number of ways = Total number of ways without restrictions – number of ways where each family sit together | |
| | $= \boxed{\left(\frac{{}^{10}C_5\times{}^5C_5}{2!}\times(5-1)!\times(5-1)!\right)-(5-1)!\times(5-1)!}$ | |
| | $= 72000$ | |
| | **Method 2:**<br>Case 1: 4 from one family, 1 from other family<br>${}^5C_4\times{}^5C_1\times(5-1)!\times{}^1C_1\times{}^4C_4\times(5-1)!=14400$<br>Case 2: 3 from one family, 2 from other family<br>${}^5C_3\times{}^5C_2\times(5-1)!\times{}^2C_2\times{}^3C_3\times(5-1)!=57600$<br>Total = 14400 + 57600 = 72000 | |
| 6 | In a soccer practice, the coach instructs the players to practise their penalty kicks. A player scores if he successfully kicks a ball into the net of a goal post. The probability that a player scores on the first kick is $\frac{2}{5}$. For all the subsequent kicks, the probability of scoring on that kick will be $\frac{4}{5}$ if the player scores in the preceding kick, and probability of scoring on that kick will be $\frac{1}{6}$ if the player did not score in the preceding kick. | |
| | **(i)** Owen kicked the ball three times consecutively for his practice. Find the probability that he scored on the third kick, given that he scored only twice out of the three kicks. | [3] |
| | **(ii)** Three players each kicked the ball four times consecutively for their practices. Find the probability that one of the players scored on all four kicks, another player scored on the first kick only, while the remaining player only scored on the second and third kicks. | [3] |
| | | |
| | **Solution** | |
| | (i) P(scored on third kick \| scored on only two of the kicks) | |
| | $=\frac{\text{P(scored on third kick and scored on only two of the kicks)}}{\text{P(scored on only two of the kicks)}}$ | |
| | $=\frac{P(SS'S)+P(S'SS)}{P(SS'S)+P(S'SS)+P(SSS')}$ | |

**Commented [KSM23]: Strategy**
Many resort to slotting – Notice here if slotting is used, you can only slot in consecutive positions ABABABABAB, or BABABABABA. If randomly choose, may end up in situations like _AA_A_A_A

**Commented [KSM24]: When randomly choosing members for groups of same size n, you need to divide by n! to to remove multiple counting, as there are n! ways of arranging the n groups of the same compositions**

**Commented [KSM25]: Question Reading**
This means that conditional probability should be considered. Plenty ignored that.

$$=\frac{\left(\frac{2}{5}\times\frac{1}{5}\times\frac{1}{6}\right)+\left(\frac{3}{5}\times\frac{1}{6}\times\frac{4}{5}\right)}{\left(\frac{2}{5}\times\frac{1}{5}\times\frac{1}{6}\right)+\left(\frac{3}{5}\times\frac{1}{6}\times\frac{4}{5}\right)+\left(\frac{2}{5}\times\frac{4}{5}\times\frac{1}{5}\right)}$$

$\approx 0.593$ (3 s.f.)

(ii) Required probability $=\text{P(SSSS)}\times\text{P(SS'S'S')}\times\text{P(S'SSS')}\times 3!$

$$=\left(\frac{2}{5}\times\left(\frac{4}{5}\right)^3\right)\left(\frac{2}{5}\times\frac{1}{5}\times\left(\frac{5}{6}\right)^2\right)\left(\frac{3}{5}\times\frac{1}{6}\times\frac{4}{5}\times\frac{1}{5}\right)\times 3!$$

$= 0.00109$ (3 s.f.)

**Commented [KSM26]: Misconception**
The 4 kicks are executed consecutively by the 3 players, with 3 different outcomes. Many add up the individual probabilities for each player instead of multiplying, and without considering the random matching of 3 outcomes to the 3 players.

**7** Grade A and grade B sugar produced by a company are packed and sold in packets. The mass of both grade A and grade B sugar sold follows independent normal distributions with mean 2.05 kg. The standard deviation for the mass of a randomly chosen packet of grade A and grade B sugar are 0.025 kg and $\sigma$ kg respectively.
If the probability that the mass of a randomly chosen packet of grade B sugar being less than 2 kg is 0.01,

(i) show that the value of $\sigma$ is 0.021493 correct to 5 significant figures. [2]

It is given that the profit per kilogram of grade A and B sugar sold is 50 cents and 40 cents respectively.

(ii) Find the probability that the total profit of three randomly chosen packets of grade A sugar is higher than three times the profit of a randomly chosen packet of grade B sugar by not more than 65 cents. [3]

(iii) Two packets of grade A sugar and $n$ packets of grade B sugar are selected at random. Find the smallest value of $n$ such that the probability that the mean mass of these packets being less than 2.06 kg is at least 0.97. [3]

**Solution:**

(i) Let $X$ and $Y$ be the random variable denoting the mass of a packet of grade A and a packet of grade B sugar respectively

**Commented [ABK27]: Inadequate working**
The **problem of NOT DEFINING VARIABLES** clearly still persists. Students are to take note that defining variables clearly is **not only a requirement** but also serves **to provide clarity** for themselves in solving such a question.

$Y \sim N(2.05, \sigma^2)$

$P(Y<2)=0.01$

$\Rightarrow P(Z<\frac{2-2.05}{\sigma})=0.01$

$\Rightarrow \frac{2-2.05}{\sigma}=-2.32635$

$\Rightarrow \sigma = 0.021493$

(ii) Let $C=(50)(X_1+X_2+X_3)-3(40)Y$

$E(C)=3(50)(2.05)-3(40)(2.05)=61.5$

$\text{Var}(C)=3(50)^2(0.025^2)+(3\times 40)^2(0.021493^2)=11.33957$

Thus $C \sim N(61.5, 11.33957)$

$$P((50)(X_1 + X_2 + X_3) - 3(40)Y) \le 65)$$
$$= P(C \le 65)$$
$$= 0.85068$$
$$= 0.851 \text{ (3 s.f.)}$$

**(iii)** Let $T$ be the mean mass of two packets of grade A sugar and $n$ packets of grade B sugar.

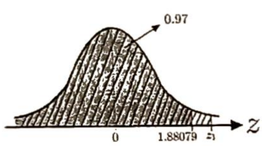
$$T = \frac{X_1 + X_2 + Y_1 + Y_2 + \ldots + Y_n}{n+2}$$

$$E(T) = 2.05 \text{ and } Var(T) = \frac{2(0.025^2) + n(0.021493^2)}{(n+2)^2}$$

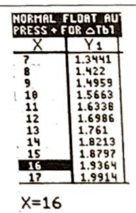
$$P(T < 2.06) \ge 0.97$$

$$\Rightarrow P\left(Z < \frac{2.06 - 2.05}{\sqrt{Var(T)}}\right) \ge 0.97$$

$$\Rightarrow P(Z < z_1) \ge 0.97$$
$$\Rightarrow z_1 > 1.88079$$




$$\Rightarrow \frac{0.01}{\sqrt{\left[\frac{2(0.025^2) + n(0.021493^2)}{(n+2)^2}\right]}} > 1.88079$$




Using GC, $n \ge 16$.
Therefore, the smallest possible value of $n$ is 16.

Alternatively,

| n | $P(T < 2.06)$ |
|---|---|
| 15 | 0.9699(< 0.97) |
| 16 | 0.9736(> 0.97) |
| 17 | 0.9768(> 0.97) |

Therefore, the smallest possible value of $n$ is 16.

**8** In a public swimming centre, the time spent by a randomly chosen user in using its facilities is $T$ minutes, is known to be normally distributed. The centre manager claims that its users spend an average of 50 minutes to use its facilities. To check this claim, time spent by a random sample of 60 users were recorded. The data recorded has an average of 47 minutes and a standard deviation of 16.4 minutes.

| | | |
|---|---|---|
| | **(i)** Find an unbiased estimate of the population variance, giving your answer correct to 2 decimal places. | [1] |
| | **(ii)** Test, at the 5% significance level, whether the centre manager overstated the average time spent. | [4] |
| | **(iii)** Another sample of size $n$ ($n > 30$) that was collected independently is now used to test, at the 5% significance level, whether the centre manager's claim is valid. For this sample, the mean time taken is 46 minutes. If the result of the test using this information and the unbiased estimate of the population variance in part **(i)** is that the null hypothesis is rejected, find the least possible value that $n$ can take. | [4] |
| | **Solution** | |
| | **(i)** Unbiased estimate of the population variance<br>$s^2 = \frac{60}{59}(16.4^2) = 273.5186441 \approx 273.52$ ( 2 decimal places) | |
| | **(ii)** Let the random variable $T$ denote the time spent in minutes using the pool facilities and $\mu$ denote the population mean time spent in minutes using the pool facilities. | |
| | To test $H_0$: $\mu = 50.0$<br>Against $H_1$: $\mu < 50.0$ (Centre manager is overstating the claim)<br>Conduct a one-tail test at 5% level of significance, i.e., $\alpha = 0.05$<br>Under $H_0$, $\bar{T} \sim N\left(50.0, \frac{273.5186441}{60}\right)$ | |
| | $\bar{t} = 47$<br>Using GC, p-value = 0.0799976609 $\approx$ 0.0800 (3 sf) | |
| | Since p-value = 0.0800 > 0.05, we do not reject $H_0$. There is insufficient evidence at 5% level of significance to conclude that that the centre manager is overstating the average time spent. | |
| | **(iii)** Using two-tailed test at 5% significance level, to reject null hypothesis, $z_{calc}$ must lie inside the critical region. | |
| | To test $H_0$: $\mu = 50.0$<br>against $H_1$: $\mu \neq 50.0$ (Centre manager's claim is valid) | |
| | Critical Region: $z \le -1.959963986$ or $z \ge 1.959963986$ | |
| | Test Statistics, $Z = \frac{\bar{T} - 50.0}{\sqrt{\frac{273.5186441}{n}}} \sim N(0, 1)$ | |
| | $\therefore z_{calc} = \frac{46.0 - 50.0}{\sqrt{\frac{273.5186441}{n}}} \le -1.959963986$ or $\frac{46.0 - 50.0}{\sqrt{\frac{273.5186441}{n}}} \ge 1.959963986$ | |

**Commented [CKJ28]: Question Reading**
Many students did not leave their final answer in 2 decimal places as required in the question.

**Commented [CKJ29]: Presentation of Working**
Many students did not define the symbols used in the question.

**Commented [CKJ30]: Common Mistakes**
Many students quoted Central Limit Theorem to be applied in their working. As the distribution of T is normally distribution, this will also imply $\bar{T}$ follow normal distribution.

**Commented [CKJ31]: Presentation of Working**
Students should define $H_0$ and $H_1$ clearly at the start of their working.

| | | |
|---|---|---|
| | $\frac{-4\sqrt{n}}{\sqrt{273.5186441}} \le -1.959963986$ or $\frac{-4\sqrt{n}}{\sqrt{273.5186441}} \ge 1.959963986$<br>$4\sqrt{n} \ge 32.41466658$ or $4\sqrt{n} \le -32.41466658$ (rejected)<br>$\sqrt{n} \ge 8.103666645$<br>$n \ge 65.669$ | |
| | Since $n$ is an integer, the least possible value of $n$ it can take is 66. | |
| | | |
| 9 | **(a)** A random variable $X$ has a binomial distribution with $n = 10$ and probability of success $p$, where $p < 0.5$.<br>**(i)** Given that $P(X = 3 \text{ or } 4) = 0.2$, write down an equation for the value of $p$, and find this value numerically. | [2] |
| | It is given that $p = \frac{1}{5}$. | |
| | **(ii)** The mean and standard deviation of $X$ are denoted by $\mu$ and $\sigma$ respectively. Find $P(\mu - \sigma < X < \mu + \sigma)$, correct to 2 decimal places. | [3] |
| | **(b)** Mr Chua attempts an online sudoku puzzle each day. The probability that he manages to solve a puzzle on any given day is 0.75, independently of any other day. | |
| | **(i)** Find the probability that he solves his third puzzle on the eighth day of his attempt. | [2] |
| | **(ii)** Find the probability that, over a period of 8 weeks, Mr Chua manages to solve at least 4 puzzles each week. | [2] |
| | | |
| | **Solution** | |
| | **(a)(i)** $X \sim B(10, p)$ | |
| | $P(X = 3 \text{ or } 4) = 0.2$ | |
| | $P(X = 3) + P(X = 4) = 0.2$ | |
| | ${}^{10}C_3 p^3(1-p)^7 + {}^{10}C_4 p^4(1-p)^6 = 0.2$ | |
| | $120p^3(1-p)^7 + 210p^4(1-p)^6 = 0.2$ | |
| | Using GC, $p = 0.570$ (rejected $\because p < 0.5$) or $p = 0.163$ | |
| | **(a) (ii)** $X \sim B(10, \frac{1}{5})$ | |
| | $\mu = E(X) = 10\left(\frac{1}{5}\right) = 2$, $\sigma^2 = 10\left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = \frac{8}{5}$ | |
| | $P(\mu - \sigma < X < \mu + \sigma)$ | |
| | $= P(2 - \sqrt{\frac{8}{5}} < X < 2 + \sqrt{\frac{8}{5}})$ | |
| | $= P(0.73509 < X < 3.2649)$ | |
| | $= P(1 \le X \le 3)$ | |
| | $= P(X \le 3) - P(X = 0)$ | |
| | $= 0.77175$ | |
| | $= 0.77$ | |
| | **(b)(i)** Let $X$ be the random variable denoting "the number of days in which Mr Chua solves the puzzle out of 7 days" | |
| | $X \sim B(7, 0.75)$ | |

**Commented [CKJ32]: Interpretation of Question**
Many students wrongly interpreted that P(X=3) = 0.2 and P(X=4) = 0.2. The lack of practice on binomial distribution questions in TYS was evident from the students' working.

**Commented [CKJ33]:** This question was well attempted. Students were familiar in solving this type of question.

**Commented [CKJ34]: Common Mistake**
Some students attempted to solve the equation algebraically. They failed to realise that they were supposed to GC to solve the equation graphically.

**Commented [CKJ35]: Common Mistake**
Many students attempted to standardize to find the probability. They failed to realise this is question on Binomial Distribution, not Normal Distribution.

Required probability $= \mathrm{P}(X=2)\times 0.75$

$= 0.00865$

**(ii)** Let $Y$ be the random variable denoting "the number of weeks in which Mr Chua solves the puzzle at least 4 times out of 8 weeks".

$Y \sim \mathrm{B}(8, \mathrm{P}(X \geq 4))$

$Y \sim \mathrm{B}(8, 0.92944)$

$\mathrm{P}(Y=8) = 0.55690 = 0.557$

Or $(0.92944)^8 = 0.55690 = 0.557$

> **Commented [CKJ36]: Presentation of Working**
> Many students did not know how to define the random variable for binomial distribution. Inappropriate letters such as Z, N were used in definition of random variables.

**10** A bag contains nine numbered discs. Three discs are numbered 3, four discs are numbered 4 and two discs are numbered $-1$. Two discs are drawn simultaneously. The sum of numbers on them, denoted by $X$, is recorded.

> **Commented [KSX37]:** Question Reading
>
> This means there is no replacement of disc. Hence total number of outcomes is NOT 81.

**(i)** Find the probability distribution for $X$. [3]

**(ii)** Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

**(iii)** Two independent observations of $X$ are taken. Find the probability that the difference between these two values is at most 5. [3]

**(iv)** Fifty independent observations of $X$ are taken. Find the approximate probability that the sum of these fifty observations is between 250 and 260. [3]

**(i)** Probability Distribution of $X$:

| $x$ | $-2$ | $2$ | $3$ | $6$ | $7$ | $8$ |
|---|---|---|---|---|---|---|
| $\mathrm{P}(X=x)$ | $\frac{2}{9}\times\frac{1}{8} = \frac{1}{36}$ | $2\times\frac{3}{9}\times\frac{2}{8} = \frac{1}{6}$ | $2\times\frac{4}{9}\times\frac{2}{8} = \frac{2}{9}$ | $\frac{3}{9}\times\frac{2}{8} = \frac{1}{12}$ | $2\times\frac{4}{9}\times\frac{3}{8} = \frac{1}{3}$ | $\frac{4}{9}\times\frac{3}{8} = \frac{1}{6}$ |

> **Commented [KSX38]:** Many students failed to consider two cases (-1,3) and (3,-1).
>
> Strategy
> Total probability should add up to 1.

**(ii)** $\mathrm{E}(X) = \left(-2\times\frac{1}{36}\right)+\left(2\times\frac{1}{6}\right)+\left(3\times\frac{2}{9}\right)+\left(6\times\frac{1}{12}\right)+\left(7\times\frac{1}{3}\right)+\left(8\times\frac{1}{6}\right)$

$= \frac{46}{9}$ or $5.1111 \approx 5.11 (3s.f.)$

$\mathrm{E}(X^2) = \left((-2)^2\times\frac{1}{36}\right)+\left(2^2\times\frac{1}{6}\right)+\left(3^2\times\frac{2}{9}\right)+\left(6^2\times\frac{1}{12}\right)+\left(7^2\times\frac{1}{3}\right)+\left(8^2\times\frac{1}{6}\right)$

$= \frac{295}{9}$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$

$= \frac{295}{9} - \left(\frac{46}{9}\right)^2$

$= \frac{539}{81}$

> **Commented [KSX39]:** Careless Mistakes
> Some students knew the formula but did not find this value correctly.

**(iii)** $\mathrm{P}(|X_1 - X_2| \leq 5) = 1 - \mathrm{P}(|X_1 - X_2| \geq 6)$

$= 1 - (2\mathrm{P}(-2, 6) + 2\mathrm{P}(-2, 7) + 2\mathrm{P}(-2, 8) + 2\mathrm{P}(2, 8))$

> **Commented [KSX40]:** Strategy
>
> Listing down all the 28 possible cases is not recommended. For those who use this method, few obtain the correct answer.
>
> Using the complementary cases to find the answer is a better strategy.

| | | | |
|---|---|---|---|
| | $= 1 - 2\times\frac{1}{36}\times\left(\frac{1}{12}+\frac{1}{3}+\frac{1}{6}\right)-2\left(\frac{1}{6}\right)^2$ | | |
| | $= \frac{197}{216}$ | | |

[Turn Over

| | | |
|---|---|---|
| | **(iv)** Since $n = 50$ is large, by Central Limit Theorem, | |
| | Let $T = X_1 + X_2 + ... + X_{50} \sim N(50\times\frac{46}{9}, 50\times\frac{539}{81})$ approximately | |
| | $T \sim N\left(\frac{2300}{9}, \frac{26950}{81}\right)$ approximately | |
| | $P(250 < T < 260) \approx 0.216$ (3 s.f.) | |
| | | |
| 11 | Research is being carried out to study the degradation of a herbicide in soil. The concentration (in percentage) of the herbicide in the soil measured over a period of 120 days is recorded. The observations are listed in the table below. It is given that one of the observations has been recorded wrongly. | |

| Number of days ($d$) | 20 | 40 | 60 | 80 | 100 | 120 |
|---|---|---|---|---|---|---|
| Concentration ($c$) | 60 | 57 | 41 | 36 | 33 | 31 |

| | | |
|---|---|---|
| | **(i)** Draw a scatter diagram to illustrate the data and circle the incorrect observation. | [3] |
| | For the rest of the question, you should exclude the incorrect observation. | |
| | **(ii)** Comment on whether a linear model would be appropriate, referring both to the scatter diagram and the context of the question. | [2] |
| | It is thought that this set of data can be modelled by one of the following formulae after removing the incorrect observation. | |
| | Model A: $c^2 = a + bd$ | |
| | Model B: $c = ae^{bd}$ | |

**Commented [KSX41]:** Misconception

Students assume $X$ follow normal distribution.

Strategy

Students know that $\overline{X}$ can be approximated to normal distribution but do not know how to proceed. Students could have considered finding $P(5 < \overline{X} < 5.2)$.

By central limit theorem, if n is large, when X follows non-normal distribution, $X_1 + X_2 + ... + X_{50}$ can be approximated to normal distribution as well.

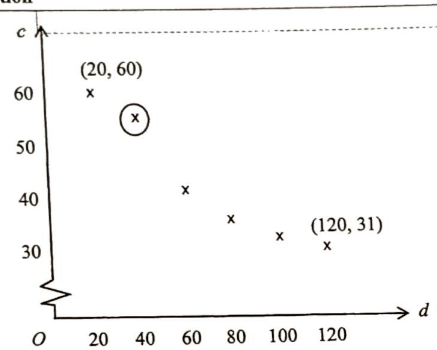
**Commented [KSX42]:** Conceptual Understanding

There is not need to do this: $P(T < 260) - P(T \le 250)$ because on the GC, the function allows you to key in lower limit and upper limit, since $T$ is a continuous random variable.

| | | |
|---|---|---|
| **(iii)** | By calculating the product moment correlation coefficients, explain clearly which of the above models is a more appropriate model for this set of data. | [3] |
| **(iv)** | Use the model you identified in **(iii)** to find the equation of a suitable regression line, and use your equation to estimate the concentration of the herbicide in the soil after 140 days. | [2] |
| **(v)** | Comment on the reliability of the estimate obtained in **(iv)**. | [1] |
| **(vi)** | Give an interpretation of the vertical intercept of the regression line obtained in **(iv)** in the context of the question. | [1] |

**Solution**

**(i)**




**Commented [SH43]: Recommendation**
1. Appropriate scale and labeling of the axes
2.Correct plot with coordinate of the end points must be labelled.
3.Correct identification of the outlier – by circling as per question requirement

Student who did not draw/ use the space well (¾ of the space provided) to draw a well scaled diagram faced difficulty in identifying the outlie correctly.
For this topic, good graphing skills is important minimize potential mistakes early.

This must be done well. Easiest way to score marks.

**(ii)** From the scatter diagram (after removing the outlier), as $d$ increases, $c$ decreases at a decreasing rate.
Also, the concentration of the herbicide will not decrease indefinitely and become a negative percentage.
Hence a linear model should not be used to model this set of data.

**Commented [SH44]: Conceptual understanding**
**Referring to the scatter diagram**

1.Describe the trend of the scatter diagram establish the difference when it is a linear model.
Many students described the scatter diagram "as d increases, c decreases.". A linear mode also described in the same manner (one with negative gradient). So, to show distinction between the 2. We should describe it as ***decreasing in a decreasing rate.***

**Referring to the context of the question**
**2.** Do not predict the trend of the scatter diagram as we do not have the data poin out of the data range. So, we should kee answers to establishing if it is a linear m then it should be decreasing indefinitely will generate negative values for concer Which is not possible based on the cont the question.

| | | |
|---|---|---|
| | (iii) Using GC, $r_A = -0.92958$ while $r_B = -0.97521$. | |
| | Since the $r$ value for model B is closer to -1 than model A, model B is more appropriate for modelling this set of data. | |
| | (iv) $c = ae^{bd}$ | |
| | $\ln c = \ln a + bd$ | |
| | From GC, $\ln c = 4.1696 - 0.0066478d$ | |
| | $\ln c = 4.16 - 0.00665d$ | |
| | When $d = 140$, $\ln c = 4.1696 - 0.0066478(140)$ | |
| | $c = 25.5059 \approx 25.5$ | |
| | (v) The estimate is unreliable because the data substituted is outside the data range [20,120] and so the linear relationship between $d$ and $\ln c$ may not hold. | |
| | (vi) Initially, the concentration of herbicides in the soil is 64.7%. | |
| | | |

**Commented [SH45]:** Question reading
For calculation of (iii)
1. Outlier/ incorrect observation must be removed before calculating the r value for each of the model.
Many students did not omit this (40,57) from their calculation thus providing incorrect $r$-values for their models. Show your GC answers up to 5sf then give final answer to 3sf.

Majority of the students did well in choosing the correct model with $|r|$ is closet to 1. Well done!

**Commented [SH46]:** Things to note
Many students did not know how to linearise ( $c = ae^{bd}$ ) to $\ln c = \ln a + bd$ . And calculating the value of $c$ posed a problem for many.

**Commented [SH47]:** Things to note
1.State the date range clearly for the examiner and add on to say that the trend may not hold and thus the estimate is not reliable.
2.Extrapolation is a process – of using a data point(out of the data range) to calculate an estimate. It doesn't warrant as an answer for marks to be awarded.

**Commented [SH48]:** Things to note
1.Initial concentration of the herbicide in percentage
2. Finding the $y-$ intercept when d=0.
3. Give your interpretation is required to show understanding. Giving answer alone is not sufficient.

# ANDERSON SERANGOON JUNIOR COLLEGE

**MATHEMATICS** **9758**

**H2 Math Prelim Paper 1 (100 marks)**

**12 Sept 2022**

**3 hours**

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS /

**READ THESE INSTRUCTIONS FIRST**

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

| Question number | Marks |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| | |
| Total | |

This document consists of 22 printed pages and 2 blank pages.

**[Turn Over**

| | | | |
|---|---|---|---|
| 1 | (i) | For positive real constant $c$, state a sequence of three transformations in terms of $c$, that will transform the graph with equation of the form $y=\mathrm{f}(2x+3)+c$ onto the graph with equation $y=\mathrm{f}(x)$. | [3] |
| | (ii) | The point with coordinates $(-2,0)$ that lies on the curve with equation of the form $y=\mathrm{f}(2x+3)+c$ is mapped onto the point with coordinates $(0,-1)$ that is on the curve with equation $y=\mathrm{f}(x)$. State the value of $c$. | [1] |

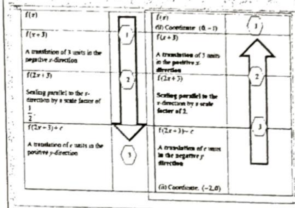
**Solution**

(i) A translation of $c$ units in the negative $y$ direction.
Scaling parallel to the $x$-axis by a scale factor of 2.
A translation of 3 units in the positive $x$-direction.

(ii) $c=1$

| | | | |
|---|---|---|---|
| $\mathrm{f}(x)$ | | $\mathrm{f}(x)$<br>(ii) Coordinate: $(0,-1)$ | 1 |
| $\mathrm{f}(x+3)$<br>A translation of 3 units in the negative $x$-direction | 1 | $\mathrm{f}(x+3)$<br>**A translation** of 3 units **in the positive $x$-direction** | |
| $\mathrm{f}(2x+3)$<br>Scaling parallel to the $x$-direction by a scale factor of $\frac{1}{2}$. | 2 | $\mathrm{f}(2x+3)$<br>**Scaling parallel to the $x$-direction by a scale factor of 2.** | 2 |
| $\mathrm{f}(2x+3)+c$<br>A translation of $c$ units in the positive $y$-direction | 3 | $\mathrm{f}(2x+3)+c$<br>**A translation of $c$ units in the negative $y$-direction**<br>(ii) Coordinate: $(-2,0)$ | 3 |

**Commented [SH1]:** Question reading / Presentation / Conceptual understanding



- Standard phrasing must be followed. Students should make it a point to memorise the phrasing for Translation / Reflection/ Scaling. Marks should not be lost in these kind of questions.

**Commented [SH2]:** Conceptual Understanding

Coordinate (-2,0) undergoes the *only 1 transformation which affects the y-axis* of which is the *Translation of c units in the negative y-direction* to Coordinate (0, -1)
Therefore: c =1

| | | | |
|---|---|---|---|
| 2 | | The complex numbers $z_1$, $z_2$ and $z_3$ are given by $z_1=\left(1-\sqrt{3}\mathrm{i}\right)^2$, $z_2=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+\mathrm{i}\sin\frac{\pi}{4}\right)\right]^6$ and $z_3=-1+\sqrt{3}\mathrm{i}$. | |
| | (i) | Using an algebraic method, find $\frac{z_2}{z_1}$ in the form $re^{\mathrm{i}\theta}$, where $r>0$ and $\theta$ is an exact real constant such that $-\pi<\theta\le\pi$. | [3] |
| | (ii) | Hence find $\frac{z_2}{z_1}+z_3$ in the form $pe^{\mathrm{i}\alpha}$, where both $r$ and $\theta$ are exact real constants such that $r>0$ and $-\pi<\theta\le\pi$. | [3] |

**Solution**

(i) $\dfrac{z_2}{z_1} = \dfrac{\left[\sqrt{2}e^{\left(\frac{\pi}{4}\right)i}\right]^6}{\left[2e^{\left(\frac{\pi}{3}\right)i}\right]^2}$

$\dfrac{z_2}{z_1} = 2e^{\left(\frac{13\pi}{6}\right)i}$

$\therefore \dfrac{z_2}{z_1} = 2e^{\left(\frac{\pi}{6}\right)i}$

(ii) $\dfrac{z_2}{z_1} + z_3 = 2e^{\left(\frac{\pi}{6}\right)i} + 2e^{\left(\frac{2\pi}{3}\right)i}$

$= 2e^{\left(\frac{\frac{\pi}{6}+\frac{2\pi}{3}}{2}\right)i}\left[e^{\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)i} + e^{\left(\frac{\frac{\pi}{6}-\frac{2\pi}{3}}{2}\right)i}\right]$

$= 2e^{\left(\frac{5\pi}{12}\right)i}\left[2\cos\left(-\frac{\pi}{4}\right)\right]$

$= 2\sqrt{2}e^{\left(\frac{5\pi}{12}\right)i}$

**Commented [LT3]:** Misconception
Did not obtain the correct exponential form for the complex numbers given.

Recommendation
1. Locate the point in the Argand Diagram before evaluating its argument.
2. Whenever possible, use exponential form to perform any simplifications. Using polar form for any simplification is strongly discourage.

**Commented [LT4]:** Question Reading
It is important to have the habit of leaving the final argument value of the complex number to be within the principal range.

**Commented [LT5]:** Misconception
Did not obtain the correct exponential form for $z_3$.

**Commented [LT6]:** Recommendation
Majority could not remember the properties learnt in the lecture. It is important to remember them.

**Commented [LT7]:** Presentation of Answer
Final answer has to be in the simplest form whenever possible.

3 It is given that the curve $C$ has equation $y = \dfrac{x^2 - x + 7}{x-2}$, $x \in \mathbb{R}$, $x \neq 2$.

**(i)** Without using a calculator, find the set of values that $y$ cannot take. [3]

**(ii)** Sketch $C$, stating clearly the equations of any asymptotes, the coordinates of the stationary points and the point(s) where the curve crosses the axes. [3]

**Solution**

(i) $y = \dfrac{x^2 - x + 7}{x-2}$

**Method 1:** $x^2 - x + 7 = y(x-2)$

$x^2 - (1+y)x + 7 + 2y = 0$

For the equation to not have real solutions, discriminant < 0

$[-(1+y)]^2 - 4(7+2y) < 0$

$y^2 - 6y - 27 < 0$

$(y-9)(y+3) < 0$

$-3 < y < 9$

$\therefore$ The set of values that $C$ cannot take is $\{y \in \mathbb{R} : -3 < y < 9\}$.

[Turn Over

**Method 2:**

$y = x + 1 + \dfrac{9}{x-2}$

$\dfrac{dy}{dx} = 1 - \dfrac{9}{(x-2)^2} = 0$ (for stationary points)

$x - 2 = 3 \text{ or } -3$

$x = 5 \text{ or } -1$

$y = 9 \text{ or } -3$

$\dfrac{d^2y}{dx^2} = \dfrac{18}{(x-2)^3}$

When x = 5, $\dfrac{d^2y}{dx^2} = \dfrac{2}{3} > 0 \Rightarrow (5,9)$ is a minimum point

When x = −1, $\dfrac{d^2y}{dx^2} = -\dfrac{2}{3} < 0 \Rightarrow (-1,-3)$ is a maximum point

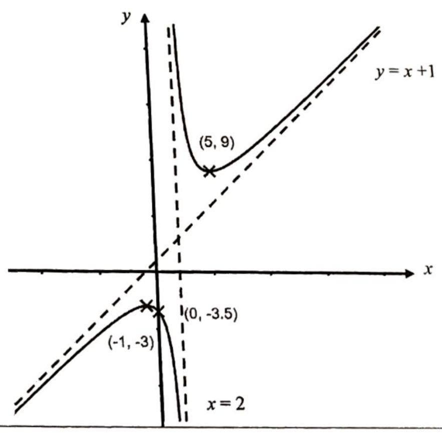
The curve is undefined at $x = 2$. For $x > 2$, $\dfrac{d^2y}{dx^2} > 0 \Rightarrow$ curve is concave upwards

For $x < 2$, $\dfrac{d^2y}{dx^2} < 0 \Rightarrow$ curve is concave downwards

Hence $y \geq 9$ or $y \leq -3$

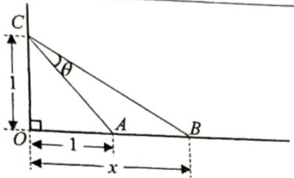
$\therefore$ The set of values that $C$ cannot take is $\{y \in \mathbb{R} : -3 < y < 9\}$

**(ii)** $y = \dfrac{x^2 - x + 7}{x-2} = x + 1 + \dfrac{9}{x-2}$




**Commented [KSM8]: Strategy**
When the differentiation method is used and ar
algebraic method is required, you must explain
the shape of the curve in all regions of x. Just
showing the existence of stationary points is
insufficient.

**4** **(i)** Show that the first two non-zero terms of the Maclaurin series for $\tan\theta$ is given by $\theta+\frac{1}{3}\theta^3$. You may use the standard results given in the List of Formulae (MF26). [2]




In the right-angle triangle $OBC$ shown above, point $A$ lies on $OB$ such that $OA=1$, $OB=x$, where $x>1$ and $OC=1$. It is given that angle $COB$ is $\frac{\pi}{2}$ radians and that angle $ACB$ is $\theta$ radians (see diagram).

**(ii)** Show that $AB=\dfrac{2\tan\theta}{1-\tan\theta}$. [2]

**(iii)** Given that $\theta$ is a sufficently small angle, show that

$$AB\approx a\theta+b\theta^2+c\theta^3$$

for exact real constants $a$, $b$ and $c$ to be determined. [3]

**Solution**

**(i)**

$$\tan\theta=\frac{\sin\theta}{\cos\theta}$$

$$=\sin\theta(\cos\theta)^{-1}$$

$$\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\frac{\theta^2}{2!}\right)^{-1}$$

$$\approx\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\frac{\theta^2}{2!}\right)$$

$$\approx\theta+\frac{\theta^3}{2!}-\frac{\theta^3}{3!}$$

$$=\theta+\frac{1}{3}\theta^3$$

**(ii)** $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}$

$$x=\frac{1+\tan\theta}{1-\tan\theta}$$

$$AB=\frac{1+\tan\theta}{1-\tan\theta}-1$$

**Commented [KSX9]:** Presentation
Use approximate sign if you are writing down first few terms of the series only.

Misconception

$$\frac{\left(\theta-\frac{\theta^3}{3!}\right)}{\left(1-\frac{\theta^2}{2!}\right)}=\theta-\frac{\theta}{3}$$

**Commented [KSX10]:** Due to lack of practice, many students did not know they have to do this step.

**Commented [KSX11]:** Method
Students can also approach this question using sine rule.

| | | |
|---|---|---|
| | $AB = \dfrac{2\tan\theta}{1-\tan\theta}$ | |
| | **(iii)** $AB = 2\tan\theta(1-\tan\theta)^{-1}$ | |
| | $\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1-\left(\theta+\dfrac{\theta^3}{3}\right)\right)^{-1}$ | |
| | $\approx 2\left(\theta+\dfrac{\theta^3}{3}\right)\left(1+\left(\theta+\dfrac{\theta^3}{3}\right)+\left(\theta+\dfrac{\theta^3}{3}\right)^2\right)$ | |
| | $\approx \left(2\theta+\dfrac{2\theta^3}{3}\right)\left(1+\theta+\theta^2\right)$<br>$\approx 2\theta+2\theta^2+\dfrac{8\theta^3}{3}$ | |
| | $a=2,\ b=2,\ c=\dfrac{8}{3}$ | |
| | | |

**Commented [KSX12]:** As the question asked to show first three term of the series, students should approximate $\tan\theta$ to $\theta+\dfrac{\theta^3}{3}$.

**Commented [KSX13]:** Presentation
Students who managed to get to the correct answer did not state the values of a, b and c.

| | | |
|---|---|---|
| **5** | **(i)** By considering $u_n-u_{n+1}$, where $u_n=\dfrac{1}{n(n+1)(n+2)}$,<br>find $\displaystyle\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$ in terms of $N$. | [3] |
| | **(ii)** Hence or otherwise, find $\displaystyle\sum_{n=5}^{N+3}\frac{1}{n(n-1)(n-2)(n-3)}$. | [3] |
| | **(iii)** Deduce that<br>$\dfrac{1}{6^2}+\dfrac{1}{12^2}+\dfrac{1}{20^2}+\dfrac{1}{30^2}+\dfrac{1}{42^2}+\ldots$<br>is less than $\dfrac{1}{18}$. Show your workings clearly. | [3] |
| | | |
| | **Solution** | |
| | **(i)** | |

**Commented [ABK14]:** Approach
When a question states "**by considering...**", we must use the approach by looking at the suggested expression $u_n-u_{n+1}$ and work from here to solve this summation problem.

**Commented [ABK15]:** Approach/ Strate
With a statement like "**hence**", typically this approach is the best way to solve the problem. The suggested "**otherwise**" approach can als used but often, it may not be the most efficie method to adopt under examination time constraint. So this has got to do with exam strategy. For the "**hence**" approach, use of t previous result is necessary.

**Commented [ABK16]:** Approach
With a statement like "**deduce**", we must st use the earlier results to prove this part of t question. We must relate clearly how this p the question use the previous results to arr the final solution.

$$u_n - u_{n+1} = \frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}$$

$$= \frac{(n+3)-n}{n(n+1)(n+2)(n+3)}$$

$$\Rightarrow u_n - u_{n+1} = \frac{3}{n(n+1)(n+2)(n+3)}$$

$$\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)}$$

$$= \frac{1}{3}\sum_{n=1}^{N}[u_n - u_{n+1}]$$

$$= \frac{1}{3}[\,u_1 - \cancel{u_2} + \cancel{u_2} - \cancel{u_3} + \cancel{u_3} - \cancel{u_4} + \cancel{\dots} + \cancel{u_{N-1}} - \cancel{u_N} + \cancel{u_N} - u_{N+1}]$$

**Alternative way (discouraged)**

$$= \frac{1}{3}\Big[\frac{1}{6} - \cancel{\frac{1}{24}} + \cancel{\frac{1}{24}} - \cancel{\frac{1}{60}} + \cancel{\frac{1}{60}} - \frac{1}{120} + \dots + \frac{1}{(N-1)(N)(N+1)} - \frac{1}{(N)(N+1)(N+2)} + \frac{1}{(N)(N+1)(N+2)} - \frac{1}{(N+1)(N+2)(N+3)}\Big]$$

Commented [ABK17]: <u>Approach</u>
In evaluating this summation using the Method of Difference, **<u>avoid substituting values</u>** for each term in this case. The approach is clear that we are using $u_n - u_{n+1}$. If we insist in evaluating each value, do note that the method is not wrong but it is **<u>NOT EFFICIENT</u>** under examination time constraint.

$$= \frac{1}{3}[u_1 - u_{N+1}]$$

$$= \frac{1}{3}\left[\frac{1}{(1)(2)(3)} - \frac{1}{(N+1)(N+2)(N+3)}\right]$$

$$= \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}$$

**(ii)** By replacing $n$ with $(n+3)$,

$$\sum_{n=5}^{N+3} \frac{1}{n(n-1)(n-2)(n-3)} = \sum_{n+3=5}^{n+3=N+3} \frac{1}{(n+3)(n+3-1)(n+3-2)(n+3-3)}$$

$$= \sum_{n=2}^{N} \frac{1}{(n)(n+1)(n+2)(n+3)}$$

$$= \sum_{n=1}^{N} \frac{1}{(n)(n+1)(n+2)(n+3)} - \sum_{n=1}^{1} \frac{1}{(n)(n+1)(n+2)(n+3)}$$

$$= \left[\frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}\right] - \left[\frac{1}{(1)(2)(3)(4)}\right]$$

$$= \frac{1}{72} - \frac{1}{3(N+1)(N+2)(N+3)}$$

**(iii)** For positive integers $n$

Commented [ABK18]: <u>Technique</u>
We have solved this in part (i) $\sum_{n=1}^{N} \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{18} - \frac{1}{3(N+1)(N+2)(N+3)}$. For part (ii), taking the "**hence**" approach, we are to use the above result.

For such a question the technique is to use "**replacement of n**". We should always start from the question that we are targeting. In this case we are "**replacing n with (n+3)**". Why "n+3"? The simple reason is that we want to make the current expression looks the same as that found in part (i) so that we can use its result.

Commented [ABK19]: <u>Continuation from above:</u>
At this juncture, the expression within the summation is now the same as that of part (i). Now, we need to split the limits to apply the result in part (i) correctly. Remember the "**cutting the cake**" method – basically counting the terms.

[Turn Over

$n^2+3n<n^2+3n+2$

$n(n+3)<(n+1)(n+2)$

$n(n+1)(n+2)(n+3)<(n+1)^2(n+2)^2$

$\therefore \dfrac{1}{n(n+1)(n+2)(n+3)}>\dfrac{1}{(n+1)^2(n+2)^2} \quad \forall n>0$

So $\displaystyle\sum_{n=1}^{N}\frac{1}{(n+1)^2(n+2)^2}<\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$

As $\displaystyle\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\ldots=\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{(n+1)^2(n+2)^2}$

$\displaystyle<\lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n(n+1)(n+2)(n+3)}$

$\displaystyle=\lim_{N\to\infty}\left[\frac{1}{18}-\frac{1}{3(N+1)(N+2)(N+3)}\right]$

$\displaystyle=\frac{1}{18}$

Thus $\displaystyle\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\ldots<\frac{1}{18}$ (deduced)

**Commented [ABK20]:** Method
This inequality must be **established** before we can proceed with the next step. **To even think** about this inequality we must first identify the expression for this sum to infinity:

$\displaystyle\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\ldots=\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)^2}$.

Looking at the RHS, that means we need to establish a link between $(n+1)^2(n+2)^2$ and $n(n+1)(n+2)(n+3)$ which is found in our original expression. This is the start of our **thinking process**.

Also, we know that we need to show $\displaystyle\frac{1}{6^2}+\frac{1}{12^2}+\frac{1}{20^2}+\ldots=\sum_{n=1}^{\infty}\frac{1}{(n+1)^2(n+2)^2}<\frac{1}{18}$.
The only $\frac{1}{18}$ that we can find is from part (i).
This would give us more clue to establish a link (inequality) between $(n+1)^2(n+2)^2$ and $n(n+1)(n+2)(n+3)$.

| | | |
|---|---|---|
| 6 | (a) Find $\displaystyle\int\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x$ for $0<x<1$. | [3] |
| | (b) (i) Sketch the graphs of $y=\lvert x^2-7\rvert$ and $y=x+5$ on the same diagram. Indicate clearly the $x$-intercepts and the values of $x$ where the two curves intersect. Hence solve the inequality $\lvert x^2-7\rvert\geq x+5$. | [4] |
| | (ii) Hence, for $a>5$, find $\displaystyle\int_3^a\left\lvert\,\lvert x^2-7\rvert-x-5\right\rvert\,\mathrm{d}x$ in terms of $a$. Leave your answer in exact form. | [3] |
| | **Solution** | |
| | **(a)** | |

**Commented [KW(W21]:** Question Reading
$\displaystyle\int\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x$ (indefinite integral) for $0<x<1$ is not equivalent to $\displaystyle\int_0^1\frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,\mathrm{d}x$ (definite integral).

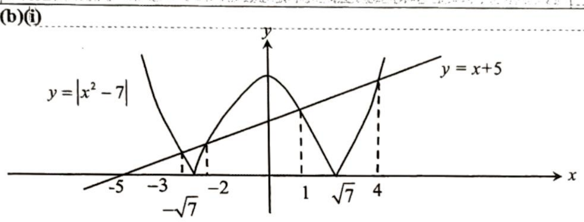
$$\int \frac{\sin^{-1}(2x-1)}{\sqrt{1-x}}\,dx = -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + \int 2\sqrt{1-x}\,\frac{2}{\sqrt{1^2-(2x-1)^2}}\,dx$$

$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + 4\int \frac{\sqrt{1-x}}{\sqrt{1-x}\sqrt{4x}}\,dx$$

$$= -2\sqrt{1-x}\cdot\sin^{-1}(2x-1) + 2\int (x)^{-\frac{1}{2}}\,dx$$

$$= -2\sqrt{1-x}\sin^{-1}(2x-1) + 4\sqrt{x} + C$$

**(b)(i)**




From the sketch, $x \le -3$ or $-2 \le x \le 1$ or $x \ge 4$

**(ii)** $\int_3^a \left| |x^2-7| - x - 5 \right| dx = \int_3^4 x+5-|x^2-7|\,dx + \int_4^a |x^2-7| - x - 5\,dx$

$= \dfrac{19}{6} + \int_4^a x^2 - 7\,dx - \int_4^a x+5\,dx$

$= \dfrac{19}{6} + \left[\dfrac{x^3}{3} - 7x\right]_4^a - \left[\dfrac{x^2}{2} + 5x\right]_4^a$

$= \dfrac{19}{6} + \left(\dfrac{a^3}{3} - 7a\right) + \dfrac{20}{3} - \left(\dfrac{a^2}{2} + 5a\right) + 28$

$= \dfrac{a^3}{3} - \dfrac{a^2}{2} - 12a + \dfrac{227}{6}$

| | | |
|---|---|---|
| 7 | A curve $C$ has parametric equations $$x = \sin^3 t\,,\quad y = \cos^2 t,\ -\frac{\pi}{2} < t < 0.$$ | |
| | The tangent at the point $P(\sin^3 p, \cos^2 p)$, $-\frac{\pi}{2} < p < 0$, meets the $x$-axis and $y$-axis at $Q$ and $R$ respectively. | |
| | **(i)** By finding the equation of the tangent at the point $P$, show that the area of the triangle $OQR$ is $-\frac{1}{12}\sin p\,(2+\cos^2 p)^2$. | [6] |
| | **(ii)** Find a cartesian equation of the locus of the mid-point of $QR$ as $p$ varies. You need not indicate its domain. | [5] |
| | | |
| | **Solution** | |

Commented [KW(W22]: Techniques
There were many mistakes observed for this part:
- Some students did not know that they need to use integration by parts.
- Others were not able to choose u and dv/dx correctly.
- Some others were not able to differentiate $\sin^{-1}(2x-1)$ and/or integrate $\frac{1}{\sqrt{1-x}}$ and/or $\frac{1}{\sqrt{x}}$ correctly.

Commented [KW(W23]: Presentation
The graph of $y = |x^2-7|$ is symmetrical about the $y$-axis. Some students could not draw the graph correctly for $x < -\sqrt{7}$ and $x > \sqrt{7}$.

Commented [KW(W24]: Concept/Presentation
Some students do not know when to use 'or' a[nd] 'and'. Others use ',' instead of 'or'.

Commented [KW(W25]: Method
Students should use part (ii) to determine how [to] split the interval from 3 to $a$ and the sign of $\left| |x^2-7| - x - 5 \right|$ in the respective intervals.

| | | |
|---|---|---|
| | **(i)** <br> $x=\sin^3 t \qquad y=\cos^2 t$ <br> $\frac{dx}{dt}=3\sin^2 t\cos t \qquad \frac{dy}{dt}=-2\sin t\cos t$ | |
| | $\frac{dy}{dx}=\frac{-2\sin t\cos t}{3\sin^2 t\cos t}=-\frac{2}{3\sin t}$ | |
| | At the point $P$, $x=\sin^3 p$ <br> $y=\cos^2 p$ <br> $\frac{dy}{dx}=-\frac{2}{3\sin p}$ | |
| | Equation of the tangent at the point $P$: <br> $y-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$ | |
| | When $y=0$, $-\cos^2 p=-\frac{2}{3\sin p}(x-\sin^3 p)$ <br> $x=\sin^3 p+\frac{3}{2}\sin p\cos^2 p$ <br> $x=\frac{1}{2}\sin p(2\sin^2 p+3\cos^2 p)$ <br> $x=\frac{1}{2}\sin p(2+\cos^2 p)$ <br> $Q\left(\frac{1}{2}\sin p(2+\cos^2 p),0\right)$ | |
| | When $x=0$, $y-\cos^2 p=-\frac{2}{3\sin p}(0-\sin^3 p)$ <br> $y=\frac{2}{3}\sin^2 p+\cos^2 p$ <br> $y=\frac{1}{3}(2\sin^2 p+3\cos^2 p)=\frac{1}{3}(2+\cos^2 p)$ <br> $R\left(0,\frac{1}{3}(2+\cos^2 p)\right)$ | |
| | Area of the triange $OQR$ <br> $=\frac{1}{2}\times OQ\times OR$ <br> $=\frac{1}{2}\left\vert\sin p(2+\cos^2 p)\right\vert\times\frac{1}{3}\left\vert 2+\cos^2 p\right\vert$ <br> $=\frac{1}{12}\left\vert\sin p\right\vert\left(2+\cos^2 p\right)^2$ <br> $=-\frac{1}{12}\sin p\left(2+\cos^2 p\right)^2 \ (\sin p<0 \because -\frac{\pi}{2}<p<0)$ | |
| | **(ii)** | |

**Commented [CKJ26]: Observation**
Many students did a conversion using double angle formula before differentiation. Some students continue to modify the expression using factor formula which was not necessary. Students should know how to differentiate the given expression directly.

**Commented [CKJ27]: Common Mistake**
Some students did not know that they had to find the gradient at the point P. Some students gave the equation of tangent at P as $y-\cos^2 p=-\frac{2}{3\sin t}(x-\sin^3 p)$ which was incorrect.

**Commented [CKJ28]: Common Mistake**
Many students did not realise the x coordinates of Q is negative or sin p < 0. To remove the modulus sign, students need to introduce a minus sign in front.

| | |
|---|---|
| | Mid point of $QR = \left(\dfrac{\frac{1}{2}\sin p(2+\cos^2 p)+0}{2}, \dfrac{0+\frac{1}{3}(2+\cos^2 p)}{2}\right)$ <br> $=\left(\frac{1}{4}\sin p(2+\cos^2 p), \frac{1}{6}(2+\cos^2 p)\right)$ |
| | $x=\frac{1}{4}\sin p(2+\cos^2 p)$ ------------ (1) <br> $y=\frac{1}{6}(2+\cos^2 p)$ ------------ (2) |
| | $\frac{(1)}{(2)}$ gives |
| | $\dfrac{x}{y}=\dfrac{\frac{1}{4}\sin p(2+\cos^2 p)}{\frac{1}{6}(2+\cos^2 p)}$ <br> $\frac{x}{y}=\frac{3}{2}\sin p$ <br> $\sin p=\frac{2x}{3y}$ |
| | $y=\frac{1}{6}(2+\cos^2 p)$ <br> $y=\frac{1}{6}\left(2+(1-\sin^2 p)\right)$ <br> $y=\frac{1}{6}\left(3-\frac{4x^2}{9y^2}\right)$ |
| | $y=\frac{1}{54y^2}\left(27y^2-4x^2\right)$ |
| | $54y^3=27y^2-4x^2$ <br> Cartesian equation of the locus of the mid-point of $QR$ is $54y^3=27y^2-4x^2$ |
| | |

**Commented [CKJ29]: Observation**
This question was poorly attempted. Many students did not attempt the question.

**Approach**
The idea is to find the mid point of QR. Exp
and y in terms of p. Then think of a way to
of the parameter p.

**8** **(a)** Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2, \quad x<0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x}, \quad x>0.$$

**(i)** Explain why the composite function gf exists. [1]

**(ii)** Find the exact value of $\mathrm{f}^{-1}\mathrm{g}^{-1}(3)$. Show your workings clearly. [3]

**(b)** For real values $a$, the function h is defined by

$$\mathrm{h}: x \mapsto ax - \frac{1}{x}, \quad x<0.$$

**(i)** If $a$ is negative, explain clearly with a well-labelled diagram, why $\mathrm{h}^{-1}$ does not exist. [4]

**(ii)** If $a = 1$, find $\mathrm{h}^{-1}$ in similar form. [3]

**Solution**

**(ai)** $\mathrm{R_f} = (0, \infty)$ and $\mathrm{D_g} = (0, \infty)$

Since $\mathrm{R_f} \subseteq \mathrm{D_g}$, the composite function gf exists.

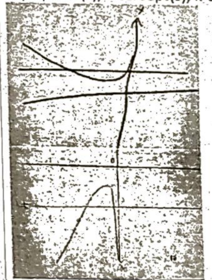
**(ii)** Let $\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$

$\mathrm{g}^{-1}(3) = \mathrm{f}(k) = k^2$ ------(1)

$\mathrm{g}(k^2) = 3$

$\frac{1}{k^2} = 3$

**Commented [LT30]:** Question Reading
Many did not comprehend what it means to be a well-labelled diagram. Some bad examples are shown below. One needs to indicate the key features of the curve like turning point(s), asymptote(s), intercept(s), if any.



**Commented [LT31]:** Presentation of Answer
It is important to tell the marker what the individual range and domain were before making the conclusion.

**Commented [LT32]:** Misconception
Some did not indicate the equal sign in this statement made.

**Commented [LT33]:** Recommendation
$\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = k$

$\Rightarrow \mathrm{f}\mathrm{f}^{-1}\mathrm{g}^{-1}(3) = \mathrm{f}(k) \quad \Rightarrow \mathrm{g}^{-1}(3) = k^2$

So one need not find the composite function to do this question.

Misconception

$\mathrm{f}^{-1}\mathrm{g}^{-1}(x) \neq (\mathrm{fg})^{-1}(x)$

In fact $\mathrm{f}^{-1}\mathrm{g}^{-1}(x) = (\mathrm{gf})^{-1}(x)$

$k=-\frac{\sqrt{3}}{3}\left(\because \mathrm{D}_{\mathrm{f}}=(-\infty,0)\right)$

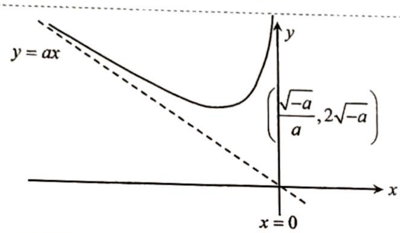
**(bi)**

$\mathrm{h}(x)=ax-\frac{1}{x}$

$\mathrm{h}'(x)=a+\frac{1}{x^2}$

For $a+\frac{1}{x^2}=0 \Rightarrow x=\frac{-1}{\sqrt{-a}}=\frac{\sqrt{-a}}{a}\ (\because x<0)$

$\therefore \mathrm{h}\left(\frac{\sqrt{-a}}{a}\right)=\sqrt{-a}+\sqrt{-a}=2\sqrt{-a}$




Since the horizontal line $y=2\sqrt{-a}+1$ cuts the curve twice, the function is not a 1-1 function and so $\mathrm{h}^{-1}$ does not exist.

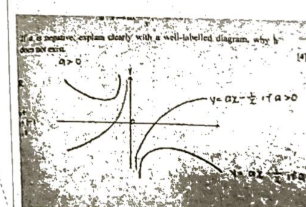
**Commented [LT34]:** Misconception
Many forgot that *k* must be in the domain o
equation (1) to work.

**Commented [LT35]:** Misconception
A number did not identify the correct *x* val
the turning point.

**Commented [LT36]:** Presentation
A number did not simplify the answer.

Note that $\frac{-a}{\sqrt{-a}}=\frac{\left(\sqrt{-a}\right)^2}{\sqrt{-a}}=\sqrt{-a}$

**Commented [LT37]:** Presentation of and Question Reading
Many failed to indicate the key features o curve (turning point, asymptotes) and sor not even draw the required curve within domain stated. One such bad example is shown below.



**Commented [LT38]:** Misconception
To disprove that it is a one-one function sufficient to suggest one particular hori line that violates the horizontal line test Many wrote "**for all lines *y=k*, k is a rea the line cuts the curve more than once** is not true for line y=0.

[Turn Over

| | | |
|---|---|---|
| | **(ii)** Let $y=\mathrm{h}(x)=x-\dfrac{1}{x}$ | |
| | $y=x-\dfrac{1}{x}$<br>$yx=x^2-1$ | |
| | $x^2-xy-1=0$ | |
| | $x=\dfrac{y\pm\sqrt{y^2+4}}{2}$ | |
| | $x=\dfrac{y-\sqrt{y^2+4}}{2}$ $(\because x<0)$ | |
| | $\mathrm{h}^{-1}: x\mapsto\dfrac{x-\sqrt{x^2+4}}{2}$, $x\in\mathbb{R}$ | |
| | | |
| 9 | **(a)** An arithmetic progression has first term $a$ and common difference $d$, where $a>0$ and $d\neq 0$. The eighth, third and second term of the progression are the first three terms of an infinite geometric progression. | |
| | **(i)** Find the common ratio of the geometric progression. | [3] |
| | **(ii)** Find the exact sum of the odd-numbered terms of the geometric progression in terms of $a$. | [3] |
| | **(b)** A programmer coded a program involving a rabbit-fox chase along a straight path to model the actual hunt for a rabbit by a fox. | |
| | The rabbit first hop is 1.75 m. In each subsequent hop, the distance covered is 1% less than its previous hop. The fox first leaps 3 m. In each subsequent leap, the distance covered is 0.02 m less than its previous leap. Initially the rabbit is 60 m ahead of the fox and assume that the rabbit and the fox start and end each hop and leap at the same time. | |
| | **(i)** By finding the total distance travelled by the fox and the rabbit after $n$ leaps and hops respectively, find the minimum number of hops and leaps for the fox to catch up with the rabbit. | [4] |
| | **(ii)** Find the number of leaps the fox takes before it comes to a stop. Hence, find the minimum starting distance, in metre, between the fox and the rabbit such that the fox will never catch up with the rabbit. Leave your answer to the nearest integer. | [2] |
| | | |
| | **Solution** | |
| | **(ai)** Let $b$ and $r$ be the first term and common ratio of the G.P. | |
| | $b=a+7d$ ---------------------- (1)<br>$br=a+2d$ ---------------------- (2)<br>$br^2=a+d$ ---------------------- (3) | |
| | From (1) and (2) gives<br>$b-br=5d$ ---------------------- (4)<br>From (2) and (3) gives<br>$br-br^2=d$ ---------------------- (5) | |
| | (4) divides (5) gives | |

**Commented [LT39]:** Misconception
A number did not select the correct equation that is based on the domain of h(x).

**Commented [LT40]:** Question Reading
A number did not express the answer in the similar form. It must be written in the form as how the question has presented.

$$\frac{1-r}{r-r^2}=5$$

$$5r^2-6r+1=0$$

$$(5r-1)(r-1)=0$$

$r=\frac{1}{5}$ or $r=1$ (rejected since $d\neq 0$)

**(ii)** From (5), $\frac{4}{25}b=d$.

And from (3), $b=-\frac{25}{3}a$

$$\left(S_\infty\right)_{\text{odd}}=\frac{b}{1-r^2}$$

$$=\frac{-\frac{25}{3}a}{1-\frac{1}{25}}$$

$$=-\frac{625a}{72}$$

**Commented [KSM41]:** **Question Reading**
Question specified that the G.P. is infinite. Many did not read this and proceed to find Sn.
**Interpretation**
Majority who got this wrong mistook the first term of G.P. to be that of the A.P.

**(bi)** $\left(S_n\right)_{\text{fox}}=\frac{n}{2}\left[2(3)+(n-1)(-0.02)\right]=n(3.01-0.01n)$

$$\left(S_n\right)_{\text{rabbit}}=\frac{1.75\left[1-0.99^n\right]}{1-0.99}=175\left(1-0.99^n\right)$$

**Commented [KSM42]:** **Strategy**
There is no need to deduce the general term from scratch here, and many wasted time to do so.

For the fox to catch the rabbit,
$n(3.01-0.01n)-175\left(1-0.99^n\right)\geq 60$

**Commented [KSM43]:** Presentation
Students should express this in inequality form to explain why the n obtained is the least.

Let $Y=n(3.01-0.01n)-175\left(1-0.99^n\right)$

From GC,

| $n$ | $Y$ |
|---|---|
| 53 | $59.171<60$ |
| 54 | $60.084>60$ |
| 55 | $60.987>60$ |

Least $n=54$

**Commented [KSM44]:** Presentation
Those who fail to show either the graph or table will not be awarded full credit.

**(ii)** Let $k$ be the starting distance between fox and rabbit.
For the fox to never catch up with the rabbit,
$k>\text{Max}\left[n(3.01-0.01n)-175\left(1-0.99^n\right)\right]$

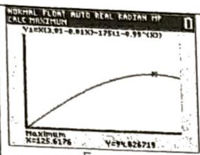
To find $n$ for which the fox stop moving, $T_n=0$

$3+(n-1)(-0.02)=0$

$n=151$

The fox takes 150 leaps before it stops moving.
From GC, for $0\leq n\leq 150$,

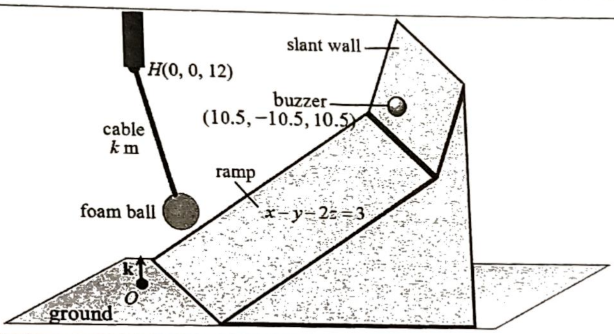
**Commented [KSM45]:** Interpretation and misconception
Here, at the 151th leap, the fox would have stopped its movement. So it's at the 150th leap before it stopped. Hence, we should analyse th graph of the difference in the distance travelle within the animals' first 150 leaps. Almost all who did this part assumed that the minimum g between the two animals occurred at the 151t or 150th leap. Do take note the Maximum or minimum point of a graph may not occur at its end points.




$k > \text{Max}\left[n(3.01-0.01n)-175\left(1-0.99^n\right)\right] = 94.827$

Hence minimum $k$ is 95 m (to the nearest integer)

**10** The production team of a popular variety show, *Sprinting Man*, is preparing a site for a segment of the show. In this segment, each participant is to sprint from the starting point, go up a ramp and press a buzzer to complete the challenge.




Referring the starting point as the origin $O$ and the horizontal ground as the $x$-$y$ plane, the top surface of the ramp has equation $x-y-2z=3$ (see diagram that is not drawn to scale). Distances are measured in metres.

(i) Find the angle of inclination of the ramp. [2]

A spherical polyurethane foam ball of radius 1 m is suspended from a point $H$ with coordinates $(0, 0, 12)$ by a cable of length $k$ m, that is taut all the time. The ball will be swung in various directions during the challenge to increase the level of difficulty.

**(ii)** If the production team wants to ensure that the foam ball will never come in contact with the ramp, find the range of values that $k$ can take. [3]

The buzzer that the participants are to press is located at the point with coordinates $(10.5, -10.5, 10.5)$. This point lies on a flat slant wall which intersects the ramp along the line $l$ with cartesian equation $x = y+20,\ z = 8.5$.

**(iii)** Find a cartesian equation of the slant wall. [3]

A camera is to be placed along a line $L$ with equation $\mathbf{r} = 12\mathbf{k} + t(\mathbf{i}+3\mathbf{j}),\ t \in \mathbb{R}$, with its position denoted by $C$.

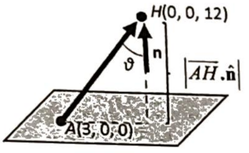
**(iv)** If the camera is at a distance of $\sqrt{254}$ m from a point $P$ with coordinates $(10, -10, 10)$, determine the possible coordinates of $C$ exactly, showing your workings. Hence deduce the point on $L$ that is nearest to $P$. [4]

**Solution**

(i) Angle of inclination of the ramp $= \cos^{-1}\dfrac{\left|\begin{pmatrix}1\\-1\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{1^2+1^2+2^2}}$

$= \cos^{-1}\dfrac{2}{\sqrt{6}} \approx 35.264^\circ = 35.3^\circ$ (1 d.p.)

(ii) A point on the ramp is $A(3, 0, 0)$. Let the normal to the ramp be $\mathbf{n} = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$.




Shortest distance from $H$ to ramp

$= |\overrightarrow{AH}.\hat{\mathbf{n}}|$

$= \dfrac{\left|\begin{pmatrix}-3\\0\\12\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-2\end{pmatrix}\right|}{\sqrt{1+1+4}} = \dfrac{27}{\sqrt{6}}$ $(\approx 11.0227)$

Since the diameter of the ball is 2 m, $0 < k < \dfrac{27}{\sqrt{6}} - 2$ [or $0 < k < 9.02$ (3 s.f.)].

**Commented [TCK46]: Question Reading**
The angle of inclination is just the angle between the two planes of the ramp and the ground. Identify the normal of these planes and take the acute angle which means a need to take the modulus value of the dot product.
Many answers were complicated and showed a lack of understanding of the question.
**Careless mistake**
The dot product and the magnitude of the normal vectors are worked wrongly.

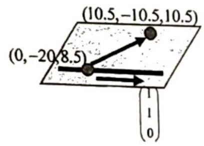
**Commented [TCK47]: Interpretation of question**
The length of cable plus the diameter of the ball (i.e. k+2) must be less than the shortest distance from H to the ramp if the ball is not to touch the ramp at all.
**Presentation**
The quality is generally poor with students not explaining what they are doing clearly and not using proper notation, e.g., taking a position vector and cartesian coordinates of a point to be the same, taking x to mean dot product.

**(iii)** $l: \mathbf{r} = \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}$.

A vector parallel to the slant wall is $\begin{pmatrix} 10.5 \\ -10.5 \\ 10.5 \end{pmatrix} - \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} = \begin{pmatrix} 10.5 \\ 9.5 \\ 2 \end{pmatrix}$.

Therefore, a normal to the slant wall is

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 10.5 \\ 9.5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$.




$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 10.5 \\ -10.5 \\ 10.5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = 31.5$

A cartesian equation of the slant wall: $4x - 4y - 2z = 63$.

**Commented [TCK48]: Presentation**

Many still write $l = \begin{pmatrix} 0 \\ -20 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and not explain what $\lambda$ is.

**Careless mistake**

Many do not convert the equation of $l$ from cartesian to vector form successfully. This affects all subsequent working.

**Commented [TCK49]: Interpretation of question**

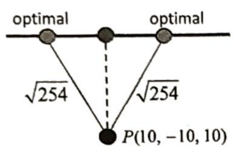
This question is about finding the equation of the plane of the slant wall. To find it, we need a point on the plane and the normal vector. We have the point which is (10.5, -10.5, 10.5) but not the normal vector. To find the normal, we need two direction vectors parallel to the plane.

**Misconception**

Many students took the normal vector of the ramp as one of the direction vectors.

**(iv)** Camera lies along the line with equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, t \in \mathbb{R}$

As the camera needs to be $\sqrt{254}$ m from $P$,




$$\sqrt{(t-10)^2 + (3t+10)^2 + (12-10)^2} = \sqrt{254}$$

$$\left.\begin{aligned} (t^2 - 20t + 100) + (9t^2 + 60t + 100) + 4 &= 254 \\ 10t^2 + 40t - 50 &= 0 \\ (t+5)(t-1) &= 0 \end{aligned}\right\}$$

$$\therefore t = 1 \text{ or } t = -5.$$

The corresponding optimal positions are

(1, 3, 12) and (−5, −15, 12).

By symmetry, the point along the line closest to $P$ is the **midpoint** of the two optimal positions. Therefore, this point has coordinates

$$\left(\frac{1-5}{2}, \frac{3-15}{2}, 12\right) = (-2, -6, 12).$$

**Commented [TCK50]: Careless mistake**
Taking $12\underset{\sim}{k}$ as $\begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix}$.

**Commented [TCK51]: Careless mistake**
Errors in expansion. Students should perhaps use the GC to solve as a safeguard.

**Commented [TCK52]: Presentation**
Not giving in coordinates form but as column vectors instead.

**Commented [TCK53]: Presentation**
Instead of mid-point theorem, many solved from scratch for the point on L nearest to P.

| | | |
|---|---|---|
| 11 | The cylindrical tank in a research laboratory has a cross-sectional area of 4 $m^2$. To cool the tank, water is pumped in and out of the tank simultaneously. The volume and height of the water in the tank at any time $t$ minutes is given by $V$ (litres) and $h$ (metres) respectively. Clean water is pumped into the tank at a rate that is proportional to $h^2$ and the water is pumped out from the tank at a rate that is proportional to $h$. | |
| | **(i)** Assume that the water does not overflow and that there is no change to the height of the water when $h$ is 10, show that $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{kh(h-10)}{4}$ where $k$ is a real constant. | [4] |
| | The tank was initially filled with clean water to a height of 2 metres. When the height of the water is 5 metres, the volume of water is increasing at a rate of 5.5 litres per minute. | |
| | **(ii)** Find the exact value of $k$. Hence find $h$ in terms of $t$. | [5] |
| | **(iii)** Sketch a graph of $h$ against $t$. Hence write down the minimum height of the cylindrical tank that will not result in the overflow of the water. | [3] |
| | | |
| | **Solution** | |
| | **(i)** <br> $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{in}}{\mathrm{d}t_{in}}-\frac{\mathrm{d}V_{out}}{\mathrm{d}t_{out}}$ <br> $\frac{\mathrm{d}V}{\mathrm{d}t}=Ah^2-Bh, \quad A,B\in\mathbb{R}$ | |
| | When $h=10$, $\frac{\mathrm{d}V}{\mathrm{d}t}=0$. | |
| | $B=10A$ | |
| | Since $V=\pi r^2h$ (and given that base area is 4 $m^2$) <br> $\therefore V=4h$ <br> $\frac{\mathrm{d}V}{\mathrm{d}t}=4\frac{\mathrm{d}h}{\mathrm{d}t}$ | |
| | $\Rightarrow 4\frac{\mathrm{d}h}{\mathrm{d}t}=Ah^2-10Ah$ | |
| | $\Rightarrow \frac{\mathrm{d}h}{\mathrm{d}t}=\frac{kh(h-10)}{4}$, where $A=k$ | |
| | **(ii)** <br> $\frac{\mathrm{d}V}{\mathrm{d}t}=5.5$ <br> $5.5=\frac{5k(5-10)}{4}\times 4$ <br> $k=-\frac{11}{50}$ | |
| | $\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{11h(h-10)}{200}$ <br> $\int\frac{1}{h^2-10h}\,\mathrm{d}h=-\int\frac{11}{200}\,\mathrm{d}t$ | |

**Commented [SH54]:** 1.<u>Question reading</u>

"Clean water is pumped into the tank at a rate that is proportional to $h^2$ and the water is pumped out from the tank at a rate that is proportional to $h$."

The above para – refers to Vol of water per unit time. Thus, the derivative $\frac{\mathrm{d}V}{\mathrm{d}t}$ should be used.

The proportionality of constant must be different for the rate of Clean water pumped in and for the rate of water pumped out. Majority of students used the same constant.

2.<u>Presentation / Conceptual understanding</u>

$\frac{\mathrm{d}h}{\mathrm{d}t}=Ah^2-Bh$ (Incorrect).

$\frac{\mathrm{d}h}{\mathrm{d}t}=0, h=10, h(Ah-B)=0$. Students used this derivative to calculate the value of B which is incorrect.

If Students are using the below expression to solve for B, then it is correct.

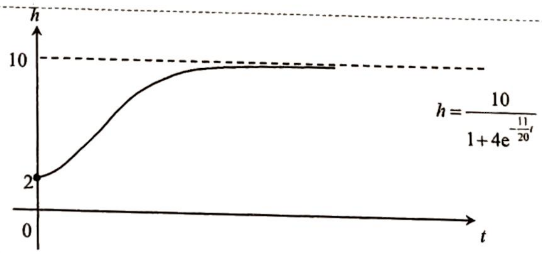
$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{\frac{\mathrm{d}V}{\mathrm{d}h}}=\frac{Ah^2-Bh}{4}$$

3.It should be $\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{h=10}=Ah^2-Bh=0$

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{h=10}=\frac{Ah^2-Bh}{4}=0$$

**Commented [SH55]:** <u>Presentation</u>

Final answer must be shown as given in the question.

| | | |
|---|---|---|
| | $\int \frac{1}{(h-5)^2-5^2}\,dh = -\frac{11}{200}t + c$ <br> $\frac{1}{10}\ln\left\lvert\frac{(h-5)-5}{h}\right\rvert = -\frac{11}{200}t + c$ | |
| | $\ln\left\lvert\frac{h-10}{h}\right\rvert = -\frac{11}{20}t + 10c$ <br> $\left\lvert 1-\frac{10}{h}\right\rvert = e^{-\frac{11}{20}t+10c}$ <br> $1-\frac{10}{h} = \pm e^{-\frac{11}{20}t+10c}$ <br> $\frac{10}{h} = 1 + Ae^{-\frac{11}{20}t}$ $\quad A = \pm e^{10c}$ <br> $h = \frac{10}{1+Ae^{-\frac{11}{20}t}}$ | |
| | When $t=0,\ h=2$ | |
| | $2 = \frac{10}{1+A}$ | |
| | $A = 4$ | |
| | $h = \frac{10}{1+4e^{-\frac{11}{20}t}}$ | |
| | **(iii)** <br>  | |
| | Minimum height of the cylindrical tank = 10 metres | |



**Commented [SH56]:** 1.<u>Conceptual mistake</u>
Many students, did not use the correct technique to integrate using Variable separable method.

Method to solve : By Partial Fractions
Or by Completing the Square

**Commented [SH57]:** <u>Presentation</u>

1.Majority of students could not simplify $\frac{h-10}{h}$ to $1-\frac{10}{h}$. If this was done earlier, then the final answer will be neat and easy to work with.
2. In (function) is defined only when the function is positive. In circumstances that you are unsure, its always safe to place the modulus. So for this question, many students omitted the modulus.
3. Removal of Modulus sign must follow through in the presentation of the answer. Many neglected this working and was penalized for not showing the removal of modulus which will manifest $\pm$ in the subsequent line.

**Commented [SH58]:** <u>Presentation of Graphs</u>
1.Initial height = 2m must be shown on the graph
2.Sketch the graph $h = \frac{10}{1+4e^{-\frac{11}{20}t}}$ to check the shape of the curve. Graph should only be drawn on the positive axis due to the context of the question.
3.Height = 10 must be captured as horizontal asymptote and graph sketched must not touch the asymptote
4. Many overlooked and lost 1 mark for not stating/ writing the min. height such that the water will not overflow.