XINMIN SECONDARY SCHOOL
新民中学
SEKOLAH MENENGAH XINMIN
Preliminary Examination 2023

CANDIDATE NAME	MARK SCHEME
CLASS	
INDEX NUMBER	

ADDITIONAL MATHEMATICS 4049/01

Paper 1 **28 August 2023**

Secondary 4 Express **2 hour 15 minutes**

Setter : Ms Vanessa Chia
Vetter : Mrs Wong Li Meng
Moderator: Ms Pang Hui Chin

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **90.**

Errors	Qn No.	Errors	Qn No.
Accuracy		Simplification	
Brackets		Units	
Geometry		***Marks Awarded***	
Presentation		***Marks Penalised***	

For Examiner's Use
90

Parent's/Guardian's Signature:

This document consists of **21** printed pages and **1** blank page.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n \ ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)...(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \tfrac{1}{2} bc \sin A$$

1 A cuboid has a square base of side $\left(2\sqrt{2}-1\right)$ cm and a volume of $\left(53-29\sqrt{2}\right)$ cm^3.

Without using a calculator, find the height of the cuboid, in cm, in the form $\left(a+b\sqrt{2}\right)$, where a and b are integers. [5]

Let h be the height of the cuboid

$\left(2\sqrt{2}-1\right)^2 \times h = 53-29\sqrt{2}$ [M1]

$$h = \frac{53-29\sqrt{2}}{\left(2\sqrt{2}-1\right)^2}$$

$$= \frac{53-29\sqrt{2}}{\left(2\sqrt{2}\right)^2 - 2\left(2\sqrt{2}\right)(1) + (1)^2}$$

$$= \frac{53-29\sqrt{2}}{8-4\sqrt{2}+1}$$

$$= \frac{53-29\sqrt{2}}{9-4\sqrt{2}} \frac{\times\left(9+4\sqrt{2}\right)}{\times\left(9+4\sqrt{2}\right)}$$

[M1- for $9-4\sqrt{2}$]
[M1- for rationalising denominator, allow ecf]

$$= \frac{\left(53-29\sqrt{2}\right)\left(9+4\sqrt{2}\right)}{(9)^2 - \left(4\sqrt{2}\right)^2}$$

$$= \frac{477+212\sqrt{2}-261\sqrt{2}-116(2)}{49}$$

[M1- for either numerator or denominator]

$$= \frac{245-49\sqrt{2}}{49}$$

$= \left(5-\sqrt{2}\right)$ cm [A1]

2 A curve is such that $\frac{\mathrm{d}y}{\mathrm{d}x} = a\mathrm{e}^{1-x} - 3x^2 + 10$, where a is a constant. The point P (1, 5) lies on the curve. The gradient of the curve at P is 12.

(a) Show that $a = 5$. [1]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\mathrm{e}^{1-x} - 3x^2 + 10$$

When $x = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 12$,

$$\left.\begin{aligned} &a\mathrm{e}^0 - 3(1)^2 + 10 = 12 \\ &a - 3 + 10 = 12 \\ &a = 5 \text{ (shown)} \end{aligned}\right\} \text{[A1]}$$

(b) Find the equation of the curve. [4]

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5\mathrm{e}^{1-x} - 3x^2 + 10$$

$$y = \int 5\mathrm{e}^{1-x} - 3x^2 + 10 \, \mathrm{d}x$$

$$\left.\begin{aligned} &= \frac{5\mathrm{e}^{1-x}}{-1} - \frac{3x^3}{3} + 10x + c \\ &= -5\mathrm{e}^{1-x} - x^3 + 10x + c \end{aligned}\right\} \begin{aligned} &\text{[M1- for } -5\mathrm{e}^{1-x} \text{]} \\ &\text{[M1- for} -x^3 + 10x \text{]} \end{aligned}$$

When $x = 1$ and $y = 5$,

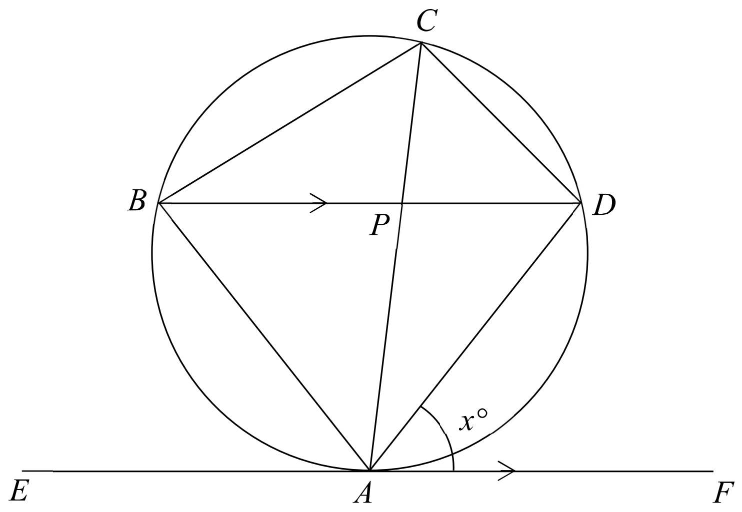
$-5\mathrm{e}^0 - (1)^3 + 10(1) + c = 5$ [M1- allow ecf]

$-5 - 1 + 10 + c = 5$

$c = 1$

$\therefore y = -5\mathrm{e}^{1-x} - x^3 + 10x + 1$ [A1]

3



A, B, C and D are points on a circle. EF is a tangent to the circle and angle $DAF = x°$. BD is parallel to EF. AC and BD intersects at P.

Prove that

(a) $AB = AD$, [3]

$\angle ADB = \angle DAF = x°$ (alt. $\angle$s, // lines) [M1- with correct reason]

$\angle ABD = \angle DAF = x°$ (alt. segment thm) [M1- with correct reason]

Since $\angle ADB = \angle ABD$, by (base $\angle$s of isos triangles), $AB = AD$.

[A1- accept "isosceles triangle"]

Deduct 1 m from overall question for incorrectly phrased reasons.

(b) AC bisects angle BCD. [2]

$\angle ACD = \angle DAF = x°$ (alt. segment thm / $\angle$s in same segment)

$\angle ACB = \angle ADB = x°$ ($\angle$s in same segment)

[M1- one correct reason]

Since $\angle ACD = \angle ACB$, $\therefore$ AC bisects $\angle BCD$.

[A1- both reasons correct + conclusion]

4 **(a)** Express 4 $\cos^2 x - 6\sin^2 x$ in the form $a\cos 2x + b$. [2]

$4\cos^2 x - 6\sin^2 x$

$= 4\left(\frac{\cos 2x + 1}{2}\right) - 6\left(\frac{1-\cos 2x}{2}\right)$ [M1- for either]

$= 2(\cos 2x + 1) - 3(1 - \cos 2x)$

$= 2\cos 2x + 2 - 3 + 3\cos 2x$

$= 5\cos 2x - 1$ [A1]

Alt mtd

$4\cos^2 x - 6\sin^2 x$

$= 4\cos^2 x - 6(\underline{1 - \cos^2 x})$ [M1]

$= 4\cos^2 x - 6 - 6\cos^2 x$

$= 10\cos^2 x - 6$

$= 10\left(\frac{\cos 2x + 1}{2}\right) - 6$

$= 5(\cos 2x + 1) - 6$

$= 5\cos 2x + 5 - 6$

$= 5\cos 2x - 1$ [A1]

(b) The equation of a curve is $y = 2x^2 - kx + 4$. Find the set of values of k for which the line $y = k - 4x$ meets the curve. [4]

$y = 2x^2 - kx + 4$ ---(1)

$y = k - 4x$ ---(2)

(1)=(2),

$2x^2 - kx + 4 = k - 4x$ [M1]

$2x^2 - kx + 4x + 4 - k = 0$

$2x^2 + (4 - k)x + (4 - k) = 0$

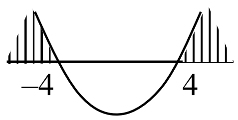
Line meets the curve: $D \geq 0$

$(4-k)^2 - 4(2)(4-k) \geq 0$ [M1]

$16 - 8k + k^2 - 32 + 8k \geq 0$

$k^2 - 16 \geq 0$ $K^2 \geqslant 16$ X -

$(k+4)(k-4) \geq 0$ [M1- for factorising]



$k \leq -4$ or $k \geq 4$ [A1]

5 A curve has the equation $y = e^{\frac{1}{2}x} + 5e^{-\frac{1}{2}x}$.

(a) Show that the exact value of the y-coordinate of the stationary point of the curve is $2\sqrt{5}$. [4]

$y = e^{\frac{1}{2}x} + 5e^{-\frac{1}{2}x}$

$\frac{dy}{dx} = e^{\frac{1}{2}x}\left(\frac{1}{2}\right) + 5e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right)$ [M1]

$= \frac{1}{2}e^{\frac{1}{2}x} - \frac{5}{2}e^{-\frac{1}{2}x}$

When $\frac{dy}{dx} = 0$, $\frac{1}{2}e^{\frac{1}{2}x} - \frac{5}{2}e^{-\frac{1}{2}x} = 0$ [M1- allow ecf]

$\frac{1}{2}e^{\frac{1}{2}x} = \frac{5}{2}e^{-\frac{1}{2}x}$

$e^{\frac{1}{2}x} = \frac{5}{e^{\frac{1}{2}x}}$

$e^{\frac{1}{2}x} \times e^{\frac{1}{2}x} = 5$

$e^x = 5$ [M1]

Mtd 1

$x = \ln 5$

$y = e^{\frac{1}{2}\ln 5} + 5e^{-\frac{1}{2}\ln 5}$

$= \left(e^{\ln 5}\right)^{\frac{1}{2}} + 5\left(e^{\ln 5}\right)^{-\frac{1}{2}}$

$= \underline{5^{\frac{1}{2}} + 5(5)^{-\frac{1}{2}}}$

$= \underline{5^{\frac{1}{2}} + 5^{\frac{1}{2}}}$

$= \sqrt{5} + \sqrt{5}$

$= \underline{2\sqrt{5}}$ (shown)

[A1]

Mtd 2

$y = \left(e^x\right)^{\frac{1}{2}} + 5\left(e^x\right)^{-\frac{1}{2}}$ ---(1)

sub into (1), $y = (5)^{\frac{1}{2}} + 5(5)^{-\frac{1}{2}}$

$= \underline{\sqrt{5} + \frac{5}{\sqrt{5}}}$

$= \sqrt{5} + \frac{5\sqrt{5}}{5}$

$= \underline{\sqrt{5} + \sqrt{5}}$

$= \underline{2\sqrt{5}}$ (shown)

[A1]

(b) Determine the nature of this stationary point. [2]

$\frac{d^2y}{dx^2} = \frac{1}{2}e^{\frac{1}{2}x}\left(\frac{1}{2}\right) - \frac{5}{2}e^{-\frac{1}{2}x}\left(-\frac{1}{2}\right)$ [M1]

$= \frac{1}{4}e^{\frac{1}{2}x} + \frac{5}{4}e^{-\frac{1}{2}x}$

When $e^x = 5$, $\frac{d^2y}{dx^2} = \frac{1}{4}(5)^{\frac{1}{2}} + \frac{5}{4}(5)^{-\frac{1}{2}}$

$= 1.11803$ or $\frac{1}{2}\sqrt{5}$

Since $\frac{d^2y}{dx^2} > 0$, it is a <u>minimum point</u>. [A1- with $\frac{d^2y}{dx^2} = 1.18$ or $\frac{d^2y}{dx^2} > 0$]

Alt mtd: First Derivative Test

$\frac{dy}{dx} = \frac{1}{2}e^{\frac{1}{2}x} - \frac{5}{2}e^{-\frac{1}{2}x}$

x	1.5	ln5	1.7
$\frac{dy}{dx}$	– ve	0	+ ve
	\	–	/

[M1]

∴ it is a minimum point. [A1]

6 The binomial expansion of $(1 + ax)^n$, where $n > 0$, in ascending powers of x is

$$1 - 6x + 36a^2x^2 + bx^3 + \dots .$$

Find the value of n, a and b. [6]

$(1 + ax)^n$

$= (1)^n + \binom{n}{1}(1)^{n-1}(ax)^1 + \binom{n}{2}(1)^{n-2}(ax)^2 + \binom{n}{3}(1)^{n-3}(ax)^3 + \dots$ [M1]

$= 1 + nax + \dfrac{n(n-1)}{2!}a^2x^2 + \dfrac{n(n-1)(n-2)}{3!}a^3x^3 + \dots$

$= 1 + nax + \dfrac{n(n-1)}{2}a^2x^2 + \dfrac{n(n-1)(n-2)}{6}a^3x^3 + \dots$

By comparing with $1 - 6x + 36a^2x^2 + bx^3 + \dots$,

$\dfrac{n(n-1)}{2}a^2x^2 = 36a^2x^2$ [M1- with $\binom{n}{2}$ expanded correctly]

$\dfrac{n(n-1)}{2} = 36$

$n(n - 1) = 72$

$n^2 - n - 72 = 0$

$\left.\begin{array}{l}(n - 9)(n + 8) = 0 \\ n = 9 \quad \text{or} \quad n = -8 \text{ (rej)}\end{array}\right\}$ [A1]

$9ax = -6x$

$a = \dfrac{-6}{9}$

$\quad = -\dfrac{2}{3}$ [A1]

$bx^3 = \dfrac{9(8)(7)}{6}\left(-\dfrac{2}{3}\right)^3 x^3$ [M1- allow ecf]

$b = -\dfrac{224}{9}$ [A1]

7 **(a)** The equation of a curve is $y = -x^2 + 10x - 17$.

(i) Express $-x^2 + 10x - 17$ in the form $a - (x + b)^2$, where a and b are constants. [2]

$-x^2 + 10x - 17$
$= -(x^2 - 10x + 17)$
$$= -\left(x^2 - 10x + \left(\frac{10}{2}\right)^2 - \left(\frac{10}{2}\right)^2 + 17\right)$$
$= -((x - 5)^2 - 8)$
$= -(x - 5)^2 + 8$
$= \underbrace{8 - (x - 5)^2}_{[M1]}$ [A1]

Alt Mtd
$a - (x + b)^2$
$= a - (x^2 + 2bx + b^2)$
$= a - x^2 - 2bx - b^2$
$= -x^2 - 2bx + (a - b^2)$
By comparing,
$-2b = 10$
$b = -5$
$a - (-5)^2 = -17$
$a - 25 = -17$
$a = 8$
$\therefore 8 - \underbrace{(x - 5)^2}_{[M1]}$ [A1]

(ii) The straight line L meets the curve at one point only. Given that L is not a tangent to the curve, what can be deduced about L? [1]

L is a vertical line. [B1]

*No marks if students wrote specific vertical line equation, e.g. $x = 1$.

(b) Express $\dfrac{3x^2 - 14x - 20}{(x-3)^2(2x+1)}$ in partial fractions. [4]

Let $\dfrac{3x^2 - 14x - 20}{(x-3)^2(2x+1)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{2x+1}$ [M1]

$3x^2 - 14x - 20 = A(x - 3)(2x + 1) + B(2x + 1) + C(x - 3)^2$
When $x = 3$,
$3(3)^2 - 14(3) - 20 = 0 + B(2(3) + 1) + 0$
$7B = -35$
$B = -5$
When $x = -\dfrac{1}{2}$,
$$3\left(-\frac{1}{2}\right)^2 - 14\left(-\frac{1}{2}\right) - 20 = 0 + 0 + C\left(-\frac{1}{2} - 3\right)^2$$
$\dfrac{49}{4}C = -\dfrac{49}{4}$
$C = -1$
When $x = 0$,
$-20 = A(-3)(1) - 5 - (-3)^2$
$-20 = -3A - 14$
$3A = 6$
$A = 2$

[M1- any two correct]
[M2- all correct]

*If partial fraction form is incorrect, **award 1m** for ability to do substitution method correctly.

$\therefore \dfrac{3x^2 - 14x - 20}{(x-3)^2(2x+1)} = \dfrac{2}{x-3} - \dfrac{5}{(x-3)^2} - \dfrac{1}{2x+1}$ [A1]

*Penalise under "presentation" if students wrote $\dfrac{2}{x-3} + \dfrac{-5}{(x-3)^2} + \dfrac{-1}{2x+1}$

8 The velocity, v m/s, of a particle travelling in a straight line, t seconds after passing through a fixed point O is given by $v = \dfrac{7}{(t+2)^3}$.

(a) Explain why the particle does not change its direction of motion. [1]

$$v = \frac{7}{(t+2)^3}$$

When $t > 0$, $\underline{(t+2)^3 > 0}$ required.

$$\frac{7}{(t+2)^3} > 0$$

Since $\underline{v > 0}$, ∴ the particle does not come to instantaneous rest and it <u>does not change its direction of motion</u>. [B1]

req'd

(b) Find the deceleration of the particle when $t = 3$. [2]

$v = 7(t+2)^{-3}$

$a = \dfrac{dv}{dt}$

$a = 7(-3)(t+2)^{-4}$ [M1]

$a = -\dfrac{21}{(t+2)^4}$

when $t = 3$, $a = -\dfrac{21}{(3+2)^4}$

$= -\dfrac{21}{625}$

∴ the deceleration is $\dfrac{21}{625}$ m/s^2. [A1- or 0.0336 m/s^2]

8 **(c)** Find the distance travelled by the particle in the third second. [4]

$v = 7(t+2)^{-3}$

Mtd 1

$s = \int 7(t+2)^{-3}\,dt$

$s = \dfrac{7(t+2)^{-2}}{-2} + c$ [M1- for $\dfrac{7(t+2)^{-2}}{-2}$]

$s = -\dfrac{7}{2(t+2)^2} + c$

when $t = 0$ and $s = 0$, $-\dfrac{7}{2(2)^2} + c = 0$ [M1 - allow ECF if incorrect integration]

$-\dfrac{7}{2(2)^2} + c = 0$

$c = \dfrac{7}{8}$

$s = -\dfrac{7}{2(t+2)^2} + \dfrac{7}{8}$

third second: from $t = 2$ to $t = 3$

When $t = 2$, $s = -\dfrac{7}{2(2+2)^2} + \dfrac{7}{8}$

$= \dfrac{21}{32}$

When $t = 3$, $s = -\dfrac{7}{2(3+2)^2} + \dfrac{7}{8}$

$= \dfrac{147}{200}$

Dist travelled in the third second

$= \dfrac{147}{200} - \dfrac{21}{32}$ [M1]

$= \dfrac{63}{800}$ m [A1- or 0.07875 m]

Mtd 2:

third second: from $t = 2$ to $t = 3$

Dist travelled in the third second

$= \int_2^3 7(t+2)^{-3}\,dx$ [M1]

$= \left[\dfrac{7(t+2)^{-2}}{-2}\right]_2^3$ [M1- for $\dfrac{7(t+2)^{-2}}{-2}$]

$= \left[-\dfrac{7}{2(t+2)^2}\right]_2^3$

$= \left(-\dfrac{7}{2(3+2)^2}\right) - \left(-\dfrac{7}{2(2+2)^2}\right)$ [M1]

$= -\dfrac{7}{50} - \left(-\dfrac{7}{32}\right)$

$= \dfrac{63}{800}$ m [A1- or 0.07875 m]

9 The equation of a curve is $y=\frac{x^2}{3x+1}$. The tangent to the curve at point P is parallel to the line $4y = x - 8$. The x-coordinate of P is negative.

(a) Find the coordinates of P. [5]

$$y=\frac{x^2}{3x+1}$$

$$\frac{dy}{dx}=\frac{(3x+1)(2x)-x^2(3)}{(3x+1)^2} \qquad \text{[M1]}$$

$$=\frac{6x^2+2x-3x^2}{(3x+1)^2}$$

$$=\frac{3x^2+2x}{(3x+1)^2} \qquad \text{[M1]}$$

$4y = x - 8$

$y=\frac{1}{4}x-2$

Gradient of tangent at $P = \frac{1}{4}$

$$\frac{3x^2+2x}{(3x+1)^2}=\frac{1}{4} \qquad \text{[M1- allow ecf]}$$

$4(3x^2 + 2x) = (3x + 1)^2$

$12x^2 + 8x = 9x^2 + 6x + 1$

$3x^2 + 2x - 1 = 0$

$(3x - 1)(x + 1) = 0$

$3x - 1 = 0$ or $x + 1 = 0$

$x = \frac{1}{3}$ (rej) $\underline{x = -1}$ [M1- for $x = -1$ with mtd to solve quad eqn]

$$y=\frac{(-1)^2}{3(-1)+1}$$

$$=-0.5$$

$\therefore P(-1, -0.5)$ [A1]

9 **(b)** The normal to the curve at P meets the y-axis at Q.

Find the area of the triangle POQ, where O is the origin. [3]

(grad normal at P) –4 [M1]

(eqn normal at P) $y = -4x + c$

When $x = -1$ and $y = -0.5$, $-0.5 = -4(-1) + c$

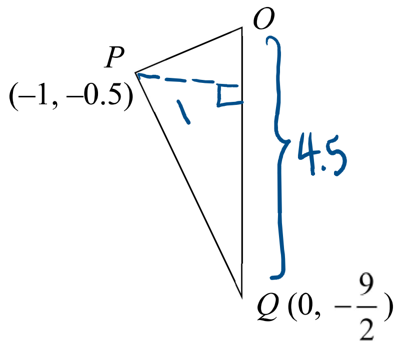
$$c = -\frac{9}{2}$$

$$y = -4x - \frac{9}{2} \qquad \text{[M1- for } -\frac{9}{2}\text{; allow ECF]}$$

$Q\ (0,\ -\frac{9}{2})$

<u>*Mtd 1*</u>

(area ΔPOQ) $\frac{1}{2} \times 4.5 \times 1 = \frac{9}{4}$ units^2 [A1- or 2.25]



<u>*Mtd 2*</u>

$$\frac{1}{2}\begin{vmatrix} 0 & -1 & 0 & 0 \\ 0 & -1.5 & -0.45 & 0 \end{vmatrix}$$

$$= \frac{1}{2}\left[(0 + 4.5 + 0) - (0 + 0 + 0)\right]$$

$$= \frac{9}{4} \text{ units}^2$$ [A1- or 2.25]

10 Water activities are held at a sports centre near Changi Beach. The depth of water, d metres, at time t hours after 0700 is modelled by $d = a \sin bt + c$, where a, b and c are positive constants.

The time between a high tide to a low tide is 6 hours. The depth of the water at high tide is 3.1 metres and the depth of the water at low tide is 0.3 metres.

(a) Show that $b = \dfrac{\pi}{6}$. [2]

Half a period of the sine graph = 6

$\dfrac{2\pi}{b} = 12$ [M1]

$2\pi = 12b$

$b = \dfrac{2\pi}{12}$

$b = \dfrac{\pi}{6}$ (shown)

[A1]

(b) Find the value of a and c. [2]

$a = \dfrac{3.1 - 0.3}{2}$

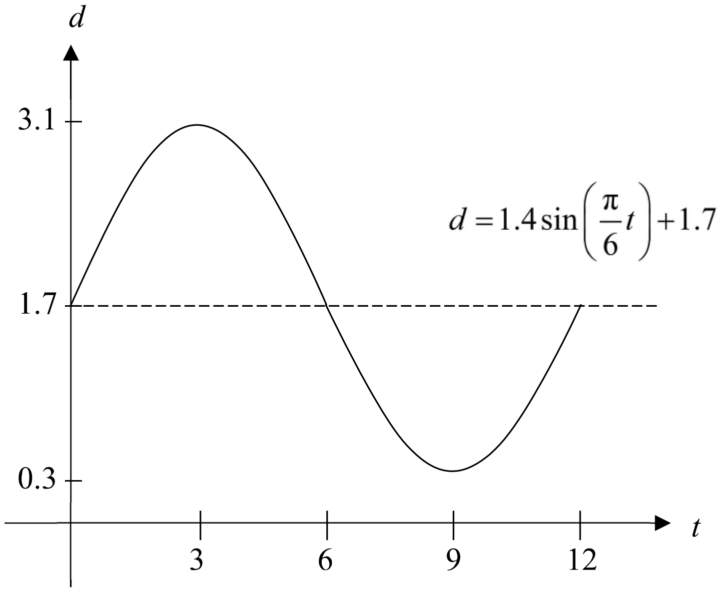
$= 1.4$ [B1]

$c = \dfrac{3.1 + 0.3}{2}$

$= 1.7$ [B1]

10 (c) Sketch the graph of d over the period 0700 to 1900 hours. [3]

$$d = 1.4\sin\left(\frac{\pi}{6}t\right) + 1.7$$

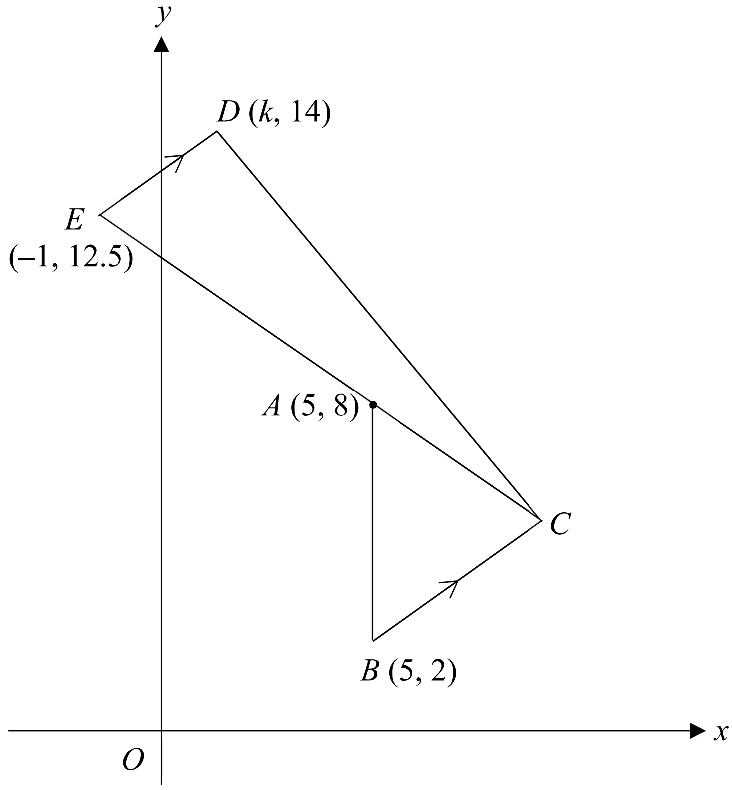


[B1- correct sine curve shape + 1 cycle]
[B1- max at $d = 3.1$, min at $d = 0.3$ and axis of curve at $d = 1.7$]
[B1- all the critical t values correct]

(d) Water activities are only permitted when the depth of water is at least a certain height. As a result, the sports centre closes x hours after 1 pm. Write an expression, in terms of x, for the number of hours that it will be closed for. [1]

Closed after 1 pm → $t > 6$
By symmetry, $6 - x - x = (6 - 2x)$ h [B1]

11



The diagram shows an isosceles triangle ABC in which the point A is (5, 8), B is (5, 2) and $AC = BC$. CA is produced to E such that ED is parallel to BC.
The point D is (k, 14) and E is (–1, 12.5). The area of triangle ABC is 12 units^2.

(a) Show that the coordinates of C is (9, 5). [2]

Let M be the midpoint of AB

$\dfrac{8+2}{2} = 5$

M (5, 5)

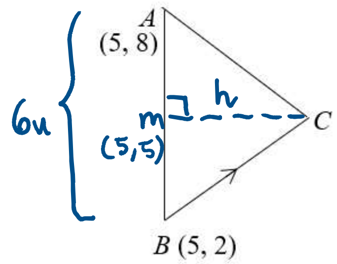
let h be the height of ΔABC

$\dfrac{1}{2} \times 6 \times h = 12$ [M1]

$h = 12 \div \dfrac{1}{2} \div 6$

$= 4$

$\therefore C$ (9, 5) (shown) [A1]



(b) Find the value of k. [3]

(grad BC) $\dfrac{5-2}{9-5} = \dfrac{3}{4}$ [M1]

(grad DE) $\dfrac{14-12.5}{k-(-1)} = \dfrac{3}{4}$ [M1- allow ecf for grad BC]

$\dfrac{1.5}{k+1} = \dfrac{3}{4}$

$3(k + 1) = 4(1.5)$
$3k + 3 = 6$
$3k = 6$
$k = 1$ (shown) [A1]

Alt Mtd

(grad BC) $\dfrac{5-2}{9-5} = \dfrac{3}{4}$ [M1]

(eqn DE) $y = \dfrac{3}{4}x + c$

When $x = -1$ and $y = 12.5$, $12.5 = \dfrac{3}{4}(-1) + c$

$c = \dfrac{53}{4}$

$y = \dfrac{3}{4}x + \dfrac{53}{4}$ [M1]

When $x = k$ and $y = 14$,

$14 = \dfrac{3}{4}k + \dfrac{53}{4}$

$\dfrac{3}{4} = \dfrac{3}{4}k$

$k = 1$ (shown) [A1]

11 (c) Prove that the angle *CDE* is not a right angle. [2]

(grad *DE*) $\frac{3}{4}$

(grad *DC*) $\frac{14-5}{1-9} = -\frac{9}{8}$

grad *DE* × grad *DC* = $\frac{3}{4} \times -\frac{9}{8}$

$= -\frac{27}{32}$ [M1]

Since grad *DE* × grad *DC* ≠ –1, ∴ the angle *CDE* is not a right angle. [A1]

Alt Mtd

$EC^2 = \left(\sqrt{(-1-9)^2 + (12.5-5)^2}\right)^2$

$= 156.25$ (or $\frac{625}{4}$)

$ED^2 + DC^2 = \left(\sqrt{(-1-1)^2 + (12.5-14)^2}\right)^2 + \left(\sqrt{(1-9)^2 + (14-5)^2}\right)^2$ [M1]

$= \frac{25}{4} + 145$

$= 151.25$ (or $\frac{605}{4}$)

Since $EC^2 \neq ED^2 + DC^2$, by the converse of Pythagoras' Theorem, the angle *CDE* is not a right angle. [A1]

(d) Find the area of triangle *CDE*. [2]

Area Δ*CDE*

$= \frac{1}{2}\begin{vmatrix} 9 & 1 & -1 & 9 \\ 5 & 14 & 12.5 & 5 \end{vmatrix}$ [M1]

$= \frac{1}{2}\left[(126 + 12.5 - 5) - (5 - 14 + 112.5)\right]$

= 15 units^2 [A1]

12 Since 1980, the number of trees in a forest has been steadily decreasing. The table shows the number of trees, N, remaining in the forest in the decades following 1980. The decade 1980 – 1989 is taken as $t = 1$, and so on.

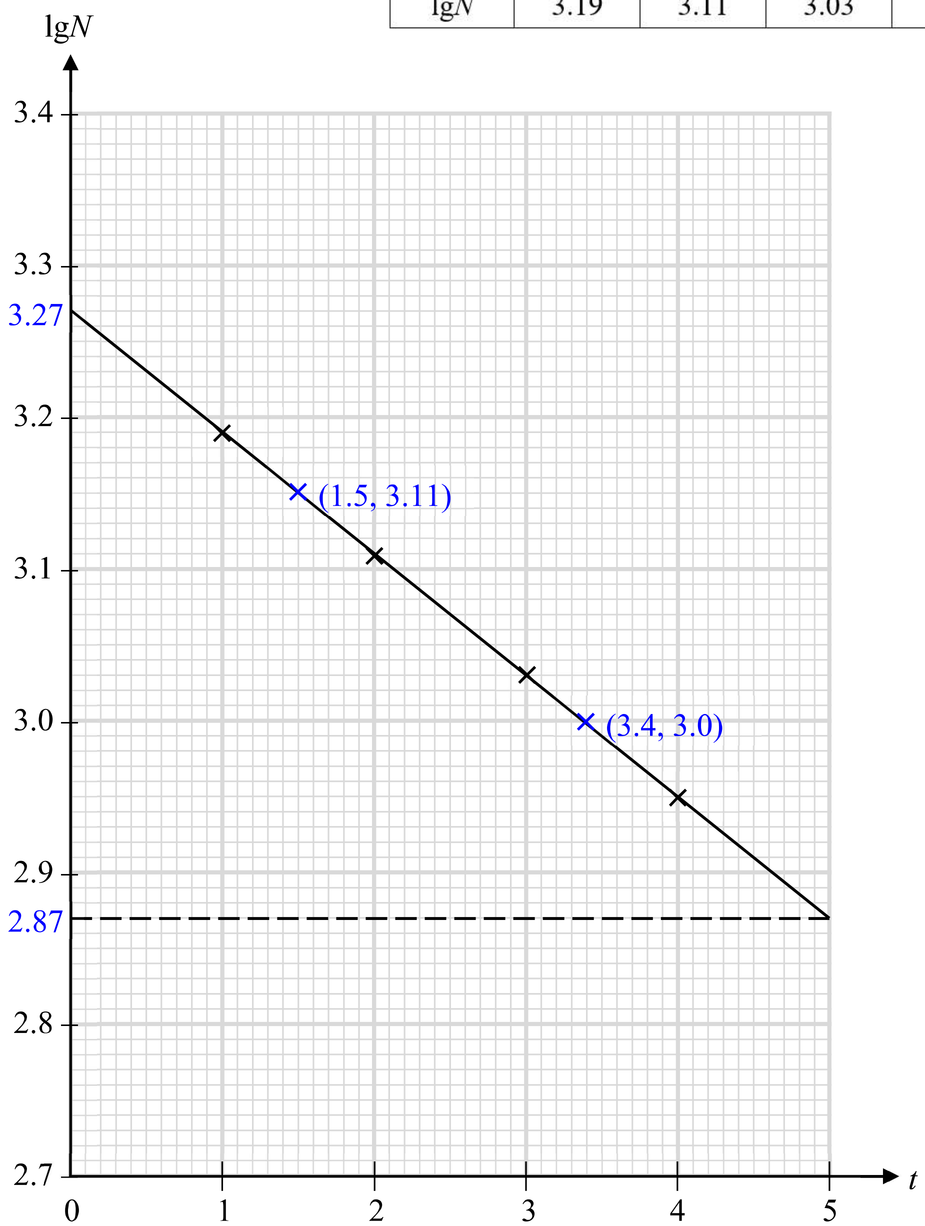
Year	1980 – 1989	1990 – 1999	2000 – 2009	2010 – 2019
Value of t	1	2	3	4
Number of trees N	1560	1300	1080	900

It is believed that these figures can be modelled by the formula $N = Ab^{-t}$, where A and b are constants.

(a) On the grid below, plot $\lg N$ against t and draw a straight line graph to illustrate the information. [2]

t	1	2	3	4
$\lg N$	3.19	3.11	3.03	2.95

(to 2dp)



[B1- correct plots]
[B1- line of best fit through $\lg N$ axis]

12 (b) Use your graph to estimate

(i) the value of A and b, [4]

$N = Ab^{-t}$
$\lg N = \lg(Ab^{-t})$
$\lg N = \lg A + \lg(b^{-t})$
$\lg N = \lg A - t\lg b$ [M1]
$\lg N = (-\lg b)t + \lg A$
(Y-int) $\lg A = 3.27$ (Accept: 3.26, 3.265, 3.27, 3.275, 3.28)
$A = 10^{3.27}$
$= 1862.08$
$= 1860$ (3sf) [A1] (Accept: 1820, 1840, 1860, 1880, 1910)

(grad) $\dfrac{3.11-3.0}{1.5-3.4} = -0.057894$

$-\lg b = -0.057894$ [M1- for $-\lg b$ = gradient,
Accept: $-0.08 \leq \text{gradient} \leq -0.05$]

$\lg b = 0.057894$
$b = 10^{0.057894}$
$= 1.1425$
$= 1.14$ (3sf) [A1- accept $10^{\text{grad within the range}}$]

(ii) the number of trees remaining in the forest in the decade 2020 – 2029. [2]

When $t = 5$,
$\lg N = 2.87$ [M1] (Accept: 2.86, 2.865, 2.87, 2.875, 2.88)
$N = 10^{2.87}$
$= 741.31$
$= 741$ (nearest integer) [A1 - accept $10^{\text{accepted } Y \text{ value}}$]

(c) Explain what A represents. [1]

$N = Ab^{-t}$
When $t = 0$, $N = A$
$\therefore$ A represents the <u>number of trees</u> in the forest <u>in 1970 – 1979</u>. [B1]

13 The equation of a curve is $y = 6x^3 + ax^2 + bx + 3$, where a and b are constants.

(a) If y is always increasing, what conditions must apply to the constants a and b? [4]

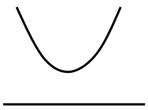
$y = 6x^3 + ax^2 + bx + 3$

$\dfrac{dy}{dx} = 6(3x^2) + 2ax + b$ [M1]

$\quad = 18x^2 + 2ax + b$

<u>Mtd 1</u>

When $\dfrac{dy}{dx} > 0$, $18x^2 + 2ax + b > 0$ [M1]



No real roots: $D < 0$

$(2a)^2 - 4(18)(b) < 0$ [M1]

$4a^2 - 72b < 0$

$4a^2 < 72b$

$a^2 < 18b$ [A1- accept equivalent ans]

<u>Mtd 2</u>

$$\frac{dy}{dx} = 18x^2 + 2ax + b$$

$$= 18\left(x^2 + \frac{a}{9}x + \frac{b}{18}\right)$$

$$= 18\left(x^2 + \frac{a}{9}x + \left(\frac{a}{18}\right)^2 - \left(\frac{a}{18}\right)^2 + \frac{b}{18}\right)$$

$$= 18\left(\left(x + \frac{a}{18}\right)^2 - \frac{a^2}{324} + \frac{b}{18}\right)$$

$$= 18\left(x + \frac{a}{18}\right)^2 - \frac{a^2}{18} + b$$ [M1]

When $\dfrac{dy}{dx} > 0$,

$$18\left(x + \frac{a}{18}\right)^2 - \frac{a^2}{18} + b > 0$$ [M1]

Since $\left(x + \dfrac{a}{18}\right)^2 \geq 0$,

$\therefore -\dfrac{a^2}{18} + b > 0$ [A1- accept equivalent ans]

13 (b) In the case where $a = 13$ and $b = -16$, find the x-coordinate of the points at which the curve intersects the line $y + 2x = 0$. [5]

Mtd 1: sub y

$y = 6x^3 + 13x^2 - 16x + 3$ ---(1)

$y + 2x = 0$

$y = -2x$ ---(2)

(1)=(2),

$6x^3 + 13x^2 - 16x + 3 = -2x$ [M1]

$6x^3 + 13x^2 - 14x + 3 = 0$

Let $f(x) = 6x^3 + 13x^2 - 14x + 3$

$f(-3) = 6(-3)^3 + 13(-3)^2 - 14(-3) + 3$

$= 0$ [M1]

$\therefore (x + 3)$ is a factor of $f(x)$

$$\begin{array}{r l}
 & 6x^2 - 5x + 1 \\
x+3 \overline{)} & 6x^3 + 13x^2 - 14x + 3 \\
 & \underline{-(6x^3 + 18x^2)} \\
 & -5x^2 - 14x \\
 & \underline{-(-5x^2 - 15x)} \\
 & x + 3 \\
 & \underline{-(x + 3)} \\
 & 0
\end{array}$$

$f(x) = (x + 3)(6x^2 - 5x + 1)$ [M1- with correct mtd shown]

when $f(x) = 0$,

$(x + 3)(6x^2 - 5x + 1) = 0$

$x + 3 = 0$ or $6x^2 - 5x + 1 = 0$

$x = -3$ $\quad (3x - 1)(2x - 1) = 0$

$3x - 1 = 0$ or $2x - 1 = 0$

$x = \frac{1}{3}$ $\quad x = \frac{1}{2}$

$\therefore x = -3$ or $\frac{1}{3}$ or $\frac{1}{2}$ [A1, A1- must show mtd to solve quadratic eqn]

Mtd 2: sub x

$y = 6x^3 + 13x^2 - 16x + 3$ ---(1)

$y + 2x = 0$

$2x = -y$

$x = -\frac{1}{2}y$ ---(2)

sub (2) into (1),

$y = 6\left(-\frac{1}{2}y\right)^3 + 13\left(-\frac{1}{2}y\right)^2 - 16\left(-\frac{1}{2}y\right) + 3$ [M1]

$y = 6\left(-\frac{1}{8}y^3\right) + 13\left(\frac{1}{4}y^2\right) + 8y + 3$

$y = -\frac{3}{4}y^3 + \frac{13}{4}y^2 + 8y + 3$

$\frac{3}{4}y^3 - \frac{13}{4}y^2 - 7y - 3 = 0$

$3y^3 - 13y^2 - 28y - 12 = 0$

Let $f(y) = 3y^3 - 13y^2 - 28y - 12$

$f(6) = 3(6)^3 - 13(6)^2 - 28(6) - 12$

$= 0$

$\therefore (y - 6)$ is a factor of $f(y)$ [M1]

$$
\begin{array}{r l}
 & 3y^2 + 5y + 2 \\
y - 6 \, \big) & 3y^3 - 13y^2 - 28y - 12 \\
 & -(3y^3 - 18y^2) \\
 & 5y^2 - 28y \\
 & -(5y^2 - 30y) \\
 & 2y - 12 \\
 & -(2y - 12) \\
 & 0
\end{array}
$$

$f(y) = (y - 6)(3y^2 + 5y + 2)$ [M1- with correct mtd shown]

when $f(y) = 0$,

$(y - 6)(3y^2 + 5y + 2) = 0$

$y - 6 = 0$ or $3y^2 + 5y + 2 = 0$

$y = 6$ $(3y + 2)(y + 1) = 0$

$3y + 2 = 0$ or $y + 1 = 0$

$y = -\frac{2}{3}$ $y = -1$

sub into (2),

$\therefore x = \underline{-3}$ or $\underline{\frac{1}{3}}$ or $\underline{\frac{1}{2}}$ [A1, A1- must show mtd to solve quadratic eqn]

*If fractional factors are used and whole qn correct, deduct 1m from the qn itself.

END OF PAPER

XINMIN SECONDARY SCHOOL
新民中学
SEKOLAH MENENGAH XINMIN
Preliminary Examinations 2023

CANDIDATE NAME Mark Scheme

CLASS

INDEX NUMBER

ADDITIONAL MATHEMATICS 4049/02

Paper 2 **29 August 2023**

Secondary 4 Express/ 5 Normal Academic **2 hour 15 minutes**

Setter : Mr Johnson Chua
Vetter : Mrs Wong Li Meng
Moderator: Ms Pang Hui Chin

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **90.**

Errors	Qn No.	Errors	Qn No.
Accuracy		Simplification	
Brackets		Units	
Geometry		*Marks Awarded*	
Presentation		*Marks Penalised*	

For Examiner's Use
90

Parent's/Guardian's Signature:

This document consists of **23** printed pages and **1** blank page.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n \ ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \tfrac{1}{2} bc \sin A$$

1 Solve the equation $36^x - 6^{x+1} = 27$. [4]

$6^{2x} - 6^x(6) = 27$

$(6^x)^2 - 6(6^x) = 27$

let $u = 6^x$

$\therefore u^2 - 6u - 27 = 0$ — (M1)

$(u-9)(u+3) = 0$.

[$u = -3$ or $u = 9$] — (M1)

$6^x = -3$ (rej.) or $6^x = 9$

$\lg 6^x = \lg 9$ — (M1)

$x = \frac{\lg 9}{\lg 6}$

$= 1.2262$

≈ 1.23 — (A1)

2 In 2015, the population of a town was estimated at 15 000.

In 2020, the numbers were estimated to have increased to 17 500.

Analysts believe that the population, N, can be modelled by the formula

$N = 15000e^{kt}$, where t is the time in years after 2015.

(a) Calculate the population of the town, to the nearest person, in 2030. [4]

$N = 15000\,e^{kt}$

$17500 = 15000\,e^{k(5)}$ — (M1)

$e^{5k} = \frac{17500}{15000}$

$5k = \ln\frac{17500}{15000}$ OR $5K = \ln\frac{7}{6}$

$k = 0.030830$ (5sf) — (M1) OR $\frac{1}{5}\ln\frac{17500}{15000}$ or $\frac{1}{5}\ln\frac{7}{6}$

$\therefore N = 15000\,e^{0.030830t}$ OR $N = 15000\,e^{\frac{t}{5}\ln\frac{17500}{15000}}$

In 2030,

$N = 15000\,e^{0.030830(15)}$ — (M1) OR $N = 15000\,e^{3\ln\frac{17500}{15000}}$

$= 23819.3$

≈ 23819 — (A1)

$\therefore$ population: 23 819

(b) The town is labelled "overcrowded" if the population exceeds 30000.

Estimate the year in which the town is first labelled "overcrowded". [2]

$$15000e^{0.030830t} = 30\,000 \quad \text{— (M1)}$$

[also accept if students use 30001 on RHS]

$$0.030830t = \ln 2$$

$$t = 22.482$$

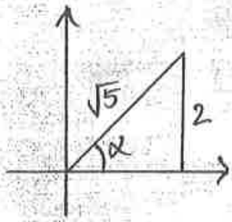
$$\approx 23$$

$$2015 + 23 = 2038$$

Ans: Year 2038 — (A1)

3 **(a)** Given that $\sin\alpha = \dfrac{2}{\sqrt{5}}$, where α is an acute angle,

show that $\sin(45° + \alpha) = \dfrac{3\sqrt{10}}{10}$. [3]



Base: $\sqrt{5-2^2}$

$= 1$

$\therefore \cos\alpha = \dfrac{1}{\sqrt{5}}$ — (M1)

$$\sin(45° + \alpha) = \sin 45° \cos\alpha + \cos 45° \sin\alpha$$

$$= \underbrace{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{5}}} + \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{5}}$$ — (M1: $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$)

$$= \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}}$$

$$= \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$ — (A1)

(b) Hence, find the value of $\cot^2(45°+\alpha)$. [3]

Recall: $1+\cot^2\theta = \operatorname{cosec}^2\theta$

$\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$\therefore \cot^2(45°+\alpha) = \operatorname{cosec}^2(45°+\alpha) - 1$ —(M1)

$\operatorname{cosec}(45°+\alpha) = \frac{10}{3\sqrt{10}}$ —(M1)

$\cot^2(45°+\alpha) = \left(\frac{10}{3\sqrt{10}}\right)^2 - 1$

$= \frac{100}{90} - 1$

$= \frac{1}{9}$ —(A1)

Alternative

$\cot^2(45°+\alpha) = \frac{\cos^2(45°+\alpha)}{\sin^2(45°+\alpha)}$

$\cos(45°+\alpha) = \cos 45° \cos\alpha - \sin 45° \sin\alpha$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{5}}\right)$

$= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} = -\frac{1}{\sqrt{10}}$ —(M1)

OR $\cos^2(45°+\alpha)$:

$= 1 - \sin^2(45°+\alpha)$

$= 1 - \left(\frac{3\sqrt{10}}{10}\right)^2$

$= \frac{1}{10}$

$\therefore \cos(45°+\alpha) = \pm\frac{1}{\sqrt{10}}$ —(M1)

$\therefore \cot^2(45°+\alpha) = \frac{\left(-\frac{1}{\sqrt{10}}\right)^2}{\left(\frac{3\sqrt{10}}{10}\right)^2}$ —(M1)

$= \frac{\left(\frac{1}{10}\right)}{\left(\frac{9}{10}\right)}$

$= \frac{1}{9}$ —(A1)

Alternative #2:

$\cot^2(45°+\alpha) = \operatorname{cosec}^2(45°+\alpha) - 1$

$= \left[\frac{1}{\sin^2(45°+\alpha)} - 1\right]$ —(M1)

$\frac{1}{\left(\frac{3\sqrt{10}}{10}\right)^2} - 1$ —(M1)

$= \frac{10}{9} - 1$

$= \frac{1}{9}$ —(A1)

[Turn over

4 **(a)** Given that $f(x) = ax^3 - 3x^2 - 2x + b$, where a and b are constants, find the values of a and b given that f(x) has a factor of $x-3$ and leaves a remainder of 4 when divided by $x+1$. [4]

$f(x) = ax^3 - 3x^2 - 2x + b$

$f(3) = 0$

$a(3)^3 - 3(3)^2 - 2(3) + b = 0$ — (M1)

$27a + b = 33$ — ①

$f(-1) = 4$

$a(-1)^3 - 3(-1)^2 - 2(-1) + b = 4$ — (M1)

$-a - 3 + 2 + b = 4$

$-a + b = 5$ — ②

① − ②: $27a - (-a) = 28$

$28a = 28$

$a = 1$ — (A1)

Sub $a = 1$ into ②: $-1 + b = 5$

$b = 6$ — (A1)

(b) (i) Factorise $250x^3 + 54y^3$. $a^3+b^3=(a+b)(a^2-ab+b^2)$ [3]

$2(125x^3+27y^3)$

$= 2[(5x)^3+(3y)^3]$ — (M1)

$= 2(5x+3y)((5x)^2+(3y)^2-5x(3y))$

$= 2\underbrace{(5x+3y)}_{(A1)}\underbrace{(25x^2-15xy+9y^2)}_{(A1)}$

(ii) Hence, given that x and y are positive integers, explain why $250x^3 + 54y^3$ is an even number. [1]

Since 2, which is an even no., is a factor of $250x^3+54y^3$, this makes $250x^3+54y^3$ an even no.

OR

Since part (i) is a multiple of 2, an even no., this makes it an even no. [OR it is even when multiplied by 2]

A1 for either.

5 **(a)** Prove the identity $\dfrac{\sin x}{\operatorname{cosec} x-1}+\dfrac{\sin x}{\operatorname{cosec} x+1}=2\tan^2 x$. [4]

LHS : $\dfrac{\sin x}{\left(\frac{1}{\sin x}-1\right)}+\dfrac{\sin x}{\left(\frac{1}{\sin x}+1\right)}$ —(M1)

$= \dfrac{\sin x}{\left(\frac{1-\sin x}{\sin x}\right)}+\dfrac{\sin x}{\left(\frac{1+\sin x}{\sin x}\right)}$

$= \dfrac{\sin^2 x}{1-\sin x}+\dfrac{\sin^2 x}{1+\sin x}$

$= \dfrac{\sin^2 x(1+\sin x)+\sin^2 x(1-\sin x)}{1-\sin^2 x}$ —(M1 for combining)

$= \dfrac{\sin^2 x\,(1+\sin x+1-\sin x)}{\cos^2 x}$ —(M1 for denominator, $1-\sin^2 x=\cos^2 x$)

$= \dfrac{\sin^2 x(2)}{\cos^2 x}$

$= 2\tan^2 x$ (A1)

Alternative

LHS: $\dfrac{(\operatorname{cosec} x+1)\cdot\sin x+(\operatorname{cosec} x-1)\cdot\sin x}{\operatorname{cosec}^2 x-1}$ —(M1 for combining fractions)

$= \dfrac{\sin x\,(\operatorname{cosec} x+1+\operatorname{cosec} x-1)}{\cot^2 x}$ —(M1 for $\operatorname{cosec}^2 x-1=\cot^2 x$)

$= \dfrac{\sin x\,(2\operatorname{cosec} x)}{\cot^2 x}$

$= \dfrac{\sin x\left(\frac{2}{\sin x}\right)}{\cot^2 x}$ —(M1 : $\operatorname{cosec} x=\frac{1}{\sin x}$)

$= \dfrac{2}{\cot^2 x} = 2\tan^2 x$ —(A1)

(b) Hence, find the exact solution(s) to the equation
$$\frac{\sin\theta}{\operatorname{cosec}\theta-1}+\frac{\sin\theta}{\operatorname{cosec}\theta+1}=5\sec\theta-4,\text{ for } -\pi\le\theta\le\pi.$$ [5]

$2\tan^2\theta = 5\sec\theta - 4$

$2\tan^2\theta - 5\sec\theta + 4 = 0$

$2(\sec^2\theta - 1) - 5\sec\theta + 4 = 0$ — (M1)

$2\sec^2\theta - 2 - 5\sec\theta + 4 = 0$

$2\sec^2\theta - 5\sec\theta + 2 = 0$

$(\sec\theta - 2)(2\sec\theta - 1) = 0$ — (M1)

$\sec\theta = 2$ or $\sec\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$ (rej) — (A1)

B.A $= \frac{\pi}{3}$ — (M1) not linked

$\theta = \underbrace{-\frac{\pi}{3}, \frac{\pi}{3}}_{(A1)}$

<u>Alt.</u> $2\tan^2\theta = 5\sec\theta - 4$

$\frac{2\sin^2\theta}{\cos^2\theta} = \frac{5}{\cos\theta} - 4$

$2\sin^2\theta = 5\cos\theta - 4\cos^2\theta$

$2(1-\cos^2\theta) = 5\cos\theta - 4\cos^2\theta$ — (M1)

$2 - 2\cos^2\theta = 5\cos\theta - 4\cos^2\theta$

$2\cos^2\theta - 5\cos\theta + 2 = 0$

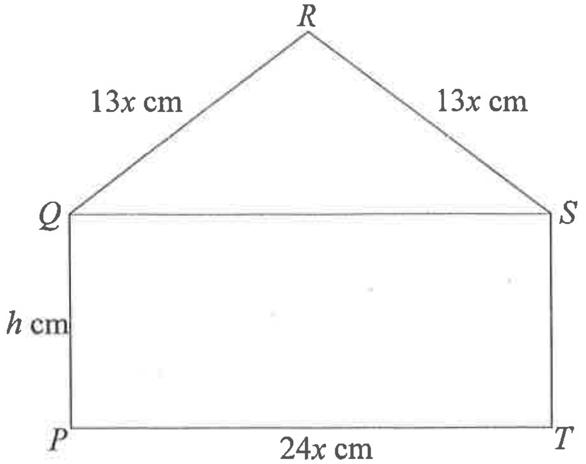
$(2\cos\theta - 1)(\cos\theta - 2) = 0$ — (M1)

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$ (rej) — (A1)

BA: $\frac{\pi}{3}$ — (M1)

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$ — (A1)

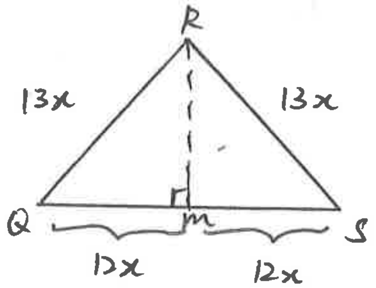
6



The diagram above shows a window frame, PQRST.
QRS is an isosceles triangle while PQST is a rectangle.
$QR = RS = 13x$ cm, $QP = h$ cm and $PT = 24x$ cm.

Given that the total perimeter of the window frame is 720 cm,

(a) show that the area of the window frame, A cm^2, is given by $A = 8640x - 540x^2$. [4]



$Rm = \sqrt{(13x)^2 - (12x)^2}$ — (M1)

$= 5x$

$2(13x) + 2(h) + 24x = 720$ — (M1)

$2h = 720 - 50x$

$h = 360 - 25x$

$A = \frac{1}{2}(24x)(5x) + (360 - 25x)(24x)$ — (M1)

$= 60x^2 + 8640x - 600x^2$

$= 8640 - 540x^2$ — (A1)

(b) Given that x can vary, an interior designer claimed that the value of h has to be 160 in order to obtain a maximum area for the window frame. Do you agree with his claim? Justify your answer with relevant workings. [5]

$$\frac{dA}{dx} = 8640 - 1080x$$

$$\frac{d^2A}{dx^2} = -1080 \ (<0)$$

Since $\frac{d^2A}{dx^2} < 0$, area is a maximum. (B1)

to find stationary value of x

$$\frac{dA}{dx} = 0$$

$\therefore 8640 - 1080x = 0$ — (M1)

$1080x = 8640$

$x = 8$ — (A1)

When $x = 8$, $h = 360 - 25(8)$ — (M1)

$= 160$

$\therefore$ I agree with the interior designer's claim that when $h = 160$, area is maximum. (A1)

Alternative

$A = 8640x - 540x^2$

$= -540(x^2 - 16x)$ — (M1)

$= -540(x^2 - 16x + 8^2) + 540(8^2)$

$= -540(x-8)^2 + 34\,560$ — (M1)

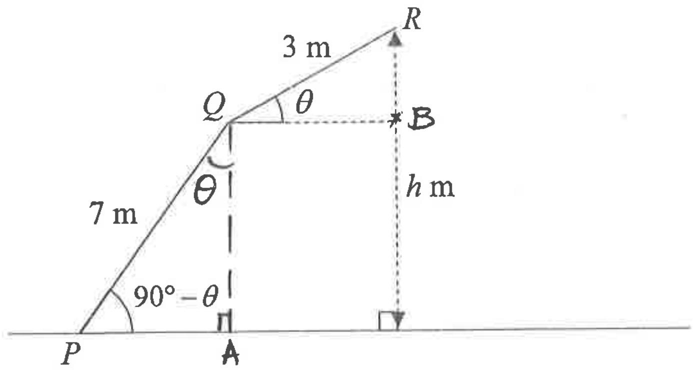
[Since coeff of x^2 is negative, A is at max. when $x = 8$] — (B1)

When $x = 8$, $h = 360 - 25(8)$ — (M1)

$= 160$.

$\therefore$ I agree with interior designer's claim that h has to be 160 to obtain maximum area. (A1)

7



The diagram shows two rods, PQ and QR, of length 7 m and 3 m respectively. PQ is hinged at P while QR is hinged at Q. The rod PQ can turn about at P and is inclined at an angle of $90° - \theta$ to the horizontal ground, where $0° \le \theta \le 90°$. The rod QR can turn about at Q in such a way that its inclination to the horizontal ground is θ. The vertical distance of R from the horizontal ground is h m.

(a) Find the values of the integers a and b for which $h = a\sin\theta + b\cos\theta$. [2]

$\angle PQA = 180° - (90° - \theta) - 90°$
$= \theta$

$h = QA + RB$

$\cos\theta = \frac{QA}{7} \Rightarrow QA = 7\cos\theta$
$\sin\theta = \frac{RB}{3} \Rightarrow RB = 3\sin\theta$ } (M1 for either)

$\therefore\ h = 3\sin\theta + 7\cos\theta$

$[\therefore a = 3,\ b = 7]$ — (A1)

(b) Using the values of a and b found in part **(a)**, express h in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

$R = \sqrt{3^2 + 7^2}$ — (M1)
$= \sqrt{58}$

$\alpha = \tan^{-1}\left(\frac{7}{3}\right)$ — (M1)
$= 66.801°$ (3dp)

$\therefore\ h = \sqrt{58}\sin(\theta + 66.801°)$
$= \sqrt{58}\sin(\theta + 66.8°)$ — (A1) (also accept if students left $\sqrt{58}$ in 3sf: 7.62)

Hence

(c) state the maximum value of h and find the corresponding value of θ, [3]

$h_{max} = \sqrt{58}$ — (A1)

$\sqrt{58}\sin(\theta + 66.801°) = \sqrt{58}$

$\sin(\theta + 66.801°) = 1$ — (M1)

$\theta + 66.801° = 90°$

$\theta = 90° - 66.801°$

$= 23.199°$

$\approx 23.2°$ (1dp) — (A1)

(d) find the value of h when PQ is inclined at an angle of 35° to the horizontal. [2]

$90° - \theta = 35°$

$\theta = 55°$

$\therefore h = \underbrace{7\cos 55° + 3\sin 55°}$ OR $\underbrace{\sqrt{58}\sin(55° + 66.801°)}$

[M1 for subst. $\theta = 55°$].

$= 6.4724$ (5sf)

≈ 6.47 (3sf) — (A1)

8 **(a)** The curve $y = kx^n$, where k and n are constants, passes through the points $(2, 64)$, $(3, 486)$ and $(a, \frac{1}{512})$. Find the values of k, n and a. [5]

Sub (2,64) and (3,486)

$64 = k(2)^n$ — ①

$486 = k(3)^n$ — ②

(M1 for either)

$\frac{①}{②}$: $\frac{64}{486} = \frac{2^n}{3^n}$ — (M1)

$\frac{32}{243} = \left(\frac{2}{3}\right)^n$

$\left(\frac{2}{3}\right)^5 = \left(\frac{2}{3}\right)^n$

$\therefore n = 5$ — (A1)

Alternative

$\lg\left(\frac{32}{243}\right) = \lg\left(\frac{2}{3}\right)^n$

$n = \lg\left(\frac{32}{243}\right) \div \lg\left(\frac{2}{3}\right)$

$= 5$.

Alternative #2

from ①: $k = \frac{64}{2^n}$ — ③

Sub ③ into ②: $486 = \frac{64}{2^n} \cdot 3^n$ — (M1)

$\frac{486}{64} = \left(\frac{3}{2}\right)^n$

$\frac{243}{32} = \left(\frac{3}{2}\right)^n$

$\left(\frac{3}{2}\right)^5 = \left(\frac{3}{2}\right)^n$

$\therefore n = 5$ — (A1)

Sub $n=5$ into ①: $64 = k(2)^5$

$k = 2$ — (A1)

$\therefore y = 2x^5 \Rightarrow \frac{1}{512} = 2a^5$

$a^5 = \frac{1}{1024}$

$a = \frac{1}{4}$ — (A1)

(b) Solve the equation $\log_3 y - \log_y 9 = 1$. [5]

$\log_3 y - \dfrac{\log_3 9}{\log_3 y} = 1$ — (M1)

$\log_3 y - \dfrac{2}{\log_3 y} = 1 \longrightarrow \underline{(\log_3 y)^2} - 2 = \log_3 y$

$2\log_3 y \neq \underline{(\log_3 y)^2}$

$2\log_3 y = \log_3 y^2$

let $x = \log_3 y$

$\therefore x - \dfrac{2}{x} = 1$

$x^2 - 2 = x$

$x^2 - x - 2 = 0$ — (M1)

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$ — (M1)

$\therefore \log_3 y = 2$ or $\log_3 y = -1$

$y = 3^2$ or $y = 3^{-1}$

$\underbrace{= 9}_{(A1)}$ $\qquad \underbrace{= \frac{1}{3}}_{(A1)}$

9 **(a)** Differentiate xe^{2x} with respect to x. [2]

$$\frac{d}{dx}\left(xe^{2x}\right)$$

$$= e^{2x} + x(2e^{2x}) \quad \text{— (M1)}$$

$$= e^{2x} + 2xe^{2x} \quad \text{— (A1)}$$

(b) A particle moves along the curve $y = xe^{2x}$ in such a way that the y-coordinate is decreasing at a constant rate of 0.2 units per second. Find the rate of change of the x-coordinate at the point where $x = -1$. [2]

$$\frac{dy}{dt} = -0.2$$

$$\text{at } x = -1, \quad \frac{dy}{dx} = e^{2(-1)} + 2(-1)e^{2(-1)}$$

$$= e^{-2} - 2e^{-2}$$

$$= -e^{-2}$$

$$\therefore \; -0.2 = -e^{-2}\left(\frac{dx}{dt}\right) \quad \text{— (M1)}$$

[Award M1 even if students sub $\frac{dy}{dt} = 0.2$]

$$\frac{dx}{dt} = \frac{-0.2}{-e^{-2}}$$

$$= 1.4778$$

$$\approx 1.48 \text{ (3sf)}$$ (also accept $0.2e^2$ or $\frac{e^2}{5}$) — (A1)

(c) Use your answer from part **(a)** to show that $\int_0^2 xe^{2x}\,dx = \dfrac{3e^4+1}{4}$ [4]

$$\int_0^2 e^{2x} + 2xe^{2x}\,dx = \left[xe^{2x}\right]_0^2 \quad \text{— (M1)}$$

$$\int_0^2 2xe^{2x}\,dx = \left[xe^{2x}\right]_0^2 - \int_0^2 e^{2x}\,dx$$

$$= 2e^4 - \left[\frac{e^{2x}}{2}\right]_0^2 \quad \text{— (M1 for } \int e^{2x}\,dx\text{)}$$

$$\int_0^2 2xe^{2x}\,dx = 2e^4 - \left(\frac{e^4}{2} - \frac{1}{2}\right) \quad \text{— (M1 for subst. of integrals)}$$

$$= 2e^4 - \frac{e^4}{2} + \frac{1}{2}$$

$$= \frac{3e^4+1}{2}$$

$$2\int_0^2 xe^{2x}\,dx = \frac{3e^4+1}{2}$$

$$\int_0^2 xe^{2x}\,dx = \frac{3e^4+1}{4} \quad \text{— (A1)}$$

Alternative

$$\int e^{2x} + 2xe^{2x}\,dx = xe^{2x} + c \quad \text{— (M1)}$$

$$\int 2xe^{2x}\,dx = xe^{2x} - \int e^{2x}\,dx + c$$

$$= xe^{2x} - \frac{e^{2x}}{2} + d \quad \text{— (M1)}$$

$$\int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + e$$

$$\int_0^2 xe^{2x} = \left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_0^2$$

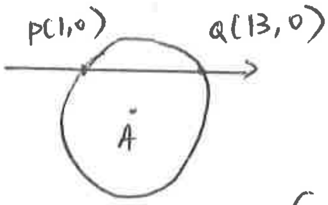
$$= \left(\frac{2e^4}{2} - \frac{e^4}{4}\right) - \left(-\frac{e^0}{4}\right) \quad \text{— (M1)}$$

$$= \frac{3e^4}{4} + \frac{1}{4}$$

$$= \frac{3e^4+1}{4} \quad \text{— (A1)}$$

10 A circle, C, of centre A intersects the x-axis at points $P(1,0)$ and $Q(13,0)$.

(a) Given that the y-coordinate of A is negative and that the radius of C_1 is $\sqrt{61}$, show that the general form of the equation of C is $x^2+y^2-14x+10y+13=0$. [4]



x-coordinate of A: $\frac{1+13}{2}=7$ — (M1)

$\therefore A(7, y)$

$(7-1)^2+(y-0)^2 = 61$ — (M1)

$6^2+y^2 = 61$

$y^2 = 61-36$

$y^2 = 25$

$y = 5$ or -5
(rej)

$\therefore A(7,-5)$

$\therefore (x-7)^2+(y+5)^2 = (\sqrt{61})^2$ — (M1, allow ECF)

$x^2-14x+49+y^2+10y+25 = 61$

$x^2+y^2-14x+10y+13 = 0$ — (A1)

Let A be (a,b).

$\therefore AP = \sqrt{(a-1)^2+b^2}$

$AQ = \sqrt{(a-13)^2+b^2}$

$(a-1)^2+b^2 = (a-13)^2+b^2$ — (M1)

$a^2-\cancel{2ab}^{\,2a}+1 = a^2-26a+169$

$24a = 168$

$a = 7$

$\therefore \sqrt{(7-1)^2+b^2} = \sqrt{61}$ or $\sqrt{(7-13)^2+b^2} = \sqrt{61}$ — (M1)

$b^2 = 61-36$

$b^2 = 25$

$b = 5$ or -5

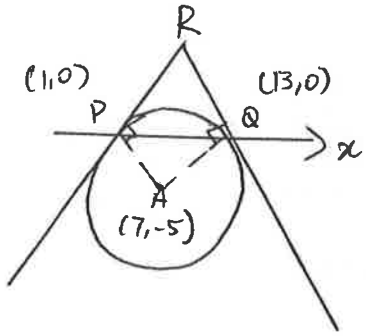
$\therefore A = (7,-5)$

$(x-7)^2+(y+5)^2 = (\sqrt{61})^2$ — (M1)

$x^2-14x+49+y^2+10y+25 = 61$

$x^2+y^2-14x+10y+13 = 0$ — (A1)

(b) R is a point which lies outside the circle.
Find the coordinates of R such that triangle APR and triangle AQR are right angle triangles which are congruent to each other. [4]



$m_{AP} = \frac{-5-0}{7-1} = \frac{-5}{6}$ $\quad$ $m_{AQ} = \frac{-5-0}{7-13} = \frac{5}{6}$

$[\, m_{PR} = \frac{6}{5}$ $\quad$ $m_{QR} = -\frac{6}{5} \,]$ — M1 for either

Eqn of PR: $y-0 = \frac{6}{5}(x-1)$ $\quad$ Eqn of QR: $y-0 = -\frac{6}{5}(x-13)$

$①: y = \frac{6}{5}x - \frac{6}{5}$ $\quad$ $②: y = -\frac{6}{5}x + \frac{78}{5}$ — M1 for either.

$\therefore$ Coordinates of R – ① = ② : $\frac{6}{5}x - \frac{6}{5} = -\frac{6}{5}x + \frac{78}{5}$ — (M1)

$\frac{12x}{5} = \frac{84}{5} \Rightarrow x = 7$

at $x = 7$, $y = \frac{36}{5}$ $\quad$ $\therefore R(7, \frac{36}{5})$ (A1)

Alternative

- M1 for findg. m_{PR} or m_{QR}
- M1 for findg Eqn of PR or Eqn of QR } Refer above.
- x-coordinate of R: $x = 7$ — (B1)

$\therefore x = 7$ — ①

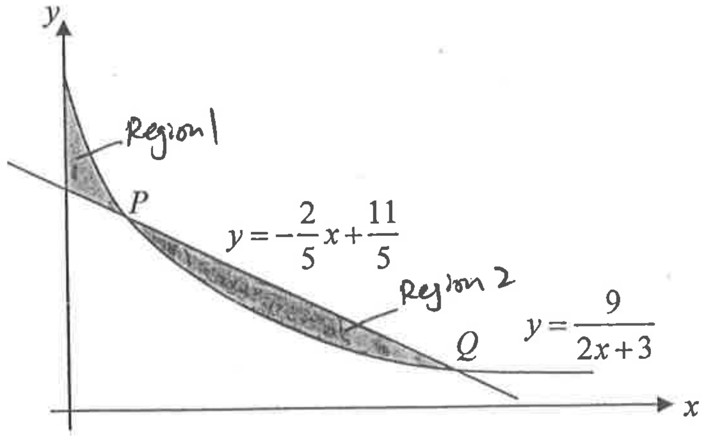
$[\, y = \frac{6}{5}x - \frac{6}{5} \;/\; y = -\frac{6}{5}x + \frac{78}{5} \,]$ — ②

Sub ① into ②: $y = \frac{36}{5}$. $\quad$ $\therefore R(7, \frac{36}{5})$ — (A1)

(c) . State the equation of another circle that passes through P and Q and has the same radius as C. [2]

$$\underbrace{(x-7)^2}_{(B1)} + \underbrace{(y-5)^2}_{(B1)} = 61$$

11



The line $y=-\frac{2}{5}x+\frac{11}{5}$ intersects the curve $y=\frac{9}{2x+3}$ at points P and Q.

Find the area of the shaded region and express it in the form $a+b\ln\frac{25}{27}$, where a and b are constants. [10]

$y=-\frac{2}{5}x+\frac{11}{5}$ — ①

$y=\frac{9}{2x+3}$ — ②

$\frac{-2x+11}{5}=\frac{9}{2x+3}$ — (M1)

$(-2x+11)(2x+3)=45$

$-4x^2-6x+22x+33=45$

$\left.\begin{array}{l}-4x^2+16x-12=0\\ x^2-4x+3=0\end{array}\right\}$ M1 for either.

$(x-1)(x-3)=0$

$[\therefore x=1 \text{ or } x=3]$ — (M1)

Continuation of working space for question 10.

Area of Region 1

$$\int_0^1 \frac{9}{2x+3}\,dx - \int_0^1 -\frac{2x}{5}+\frac{11}{5}\,dx \quad \text{— (M1)}$$

$$= \left[\frac{9\ln(2x+3)}{2}\right]_0^1 - \left[\frac{-2x^2}{5(2)}+\frac{11x}{5}\right]_0^1 \quad \text{— (M1, M1 for correct integration)}$$

$$= \frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - \left(-\frac{1}{5}+\frac{11}{5}\right)$$

$$= \frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - 2$$

Area of region 2

$$\int_1^3 -\frac{2x}{5}+\frac{11}{5}\,dx - \int_1^3 \frac{9}{2x+3}\,dx \quad \text{— (M1)}$$

$$= \left[-\frac{x^2}{5}+\frac{11x}{5}\right]_1^3 - \left[\frac{9}{2}\ln(2x+3)\right]_1^3$$

$$= \left[-\frac{9}{5}+\frac{33}{5}-\left(-\frac{1}{5}+\frac{11}{5}\right)\right] - \left(\frac{9}{2}\ln 9 - \frac{9}{2}\ln 5\right)$$

$$= \frac{24}{5} - 2 - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 5$$

total area

$$\frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - 2 + \frac{24}{5} - 2 - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 5 \quad \text{— (M1) for adding area}$$

$$= \frac{4}{5} + \frac{9}{2}\left(\ln 5 - \ln 3 - \ln 9 + \ln 5\right)$$

$$= \frac{4}{5} + \frac{9}{2}\left(\ln \frac{5\times 5}{3\times 9}\right)$$

$$= \underbrace{\frac{4}{5}}_{(A1)} + \frac{9}{2}\underbrace{\ln\left(\frac{25}{27}\right)}_{(A1)}$$

--- End of Paper ---

Q10. Alternative (after solving $x=1$ or $x=3$ on pg 22)

when $x=0$,

$$y=-\frac{2}{5}x+\frac{11}{5}$$
$$=\frac{11}{5}$$

when $x=1$, $y=\frac{9}{2+3}$
$$=\frac{9}{5} \quad \therefore P\left(1,\frac{9}{5}\right)$$

when $x=3$, $y=\frac{9}{6+3} \quad \therefore Q(3,1)$
$$=1.$$

area of region 1

$$\int_0^1 \frac{9}{2x+3}\,dx-\frac{1}{2}\left(\frac{11}{5}+\frac{9}{5}\right)(1) \text{ —— M1}$$
$$=\left[9\left(\frac{\ln(2x+3)}{2}\right)\right]_0^1-2 \text{ —— M1 integration} \text{ —— M1 trapezium area.}$$
$$=\frac{9\ln 5}{2}-\frac{9\ln 3}{2}-2$$
$$=\frac{9}{2}(\ln 5-\ln 3)-2$$
$$=\frac{9}{2}\ln\frac{5}{3}-2$$

area of region 2

$$\frac{1}{2}\left(\frac{9}{5}+1\right)(3-1)-\int_1^3\frac{9}{2x+3}\,dx \text{ —— M1}$$
$$=2\frac{4}{5}-\left[9\left(\frac{\ln(2x+3)}{2}\right)\right]_1^3$$
$$=2\frac{4}{5}-\frac{9}{2}\ln 9+\frac{9}{2}\ln 5$$
$$=2\frac{4}{5}-\frac{9}{2}(\ln 9-\ln 5)$$
$$=2\frac{4}{5}-\frac{9}{2}\ln\frac{9}{5}$$

$\therefore$ area of shaded region

$$=\left(\frac{9}{2}\ln\frac{5}{3}-2\right)+\left(2\frac{4}{5}-\frac{9}{2}\ln\frac{9}{5}\right) \text{ —— M1}$$
$$=\frac{4}{5}+\frac{9}{2}\left(\ln\frac{5}{3}-\ln\frac{9}{5}\right)$$
$$=\frac{4}{5}+\frac{9}{2}\ln\left(\frac{5}{3}\div\frac{9}{5}\right)$$
$$=\underbrace{\frac{4}{5}}_{A1}+\underbrace{\frac{9}{2}\ln\frac{25}{27}}_{A1}.$$