HILLGROVE SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2023
SECONDARY FOUR (EXPRESS)

CANDIDATE NAME	( )	CLASS	-
CENTRE NUMBER	S	INDEX NUMBER	

Additional Mathematics **4049/01**

Paper 1 **23 August 2023**

2 hours 15 minutes

Candidates answer on the Question Paper. **10.45 a.m. – 1.00 p.m.**

No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

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Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
TOTAL	**90**

Parent's/ Guardian's Signature: ___________________

Setter: Mdm Lee Li Lian

This document consists of **18** printed pages, including this page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 The equation of a curve is $y = 4x^2 - 16x + 19$.

(a) By expressing $4x^2 - 16x + 19$ in the form $a(x+b)^2 + c$, where a, b and c are constants, find the coordinates of the stationary point on the curve. [2]

(b) The line $y = 4x + 3$ intersects the curve at the points A and B. Find the value of k for which the distance AB can be expressed as $3\sqrt{k}$. [4]

2 It is given that $\left(\sqrt{5}\right)^9 - \left(\sqrt{5}\right)^7 + \left(\sqrt{5}\right)^5 - \left(\sqrt{5}\right)^3 + 105\left(\sqrt{5}\right) = 5^k$. By factorisation, find the value of k. [4]

3 The loudness of a sound can be measured using the equation $L = 10\lg\frac{I}{I_0}$, where I is the intensity of sound to be measured and I_0 is the intensity of sound that can barely be heard, also known as the threshold of hearing. The unit of L is the decibel (dB).

(a) Given that the loudness of a scream is 110 dB, find the ratio of the intensity of the scream to the threshold of hearing. [2]

(b) 130 dB is the pain threshold (the maximum level of sound we can hear without feeling intense pain and instantly damaging our hearing). [2]

Explain the impact, on hearing, the loudness of the sound of an unknown object falling from the sky onto Earth, if it has a sound intensity of $10^{-10.5}$ units and threshold of hearing of 10^{-25} units.

4 Given that the range of values of x where $x^2 + ax < b$ is $-4 < x < 5$, find the value of a and of b. [4]

5 The line $y = mx + c$ is drawn on the same axes as the curve $y = 4x - 2x^2$.

(a) Given that the line is a tangent to the curve when $c = \frac{1}{2}$, find the possible values of m. [3]

(b) The line $y = mx + c$ has a negative y-intercept. Do the line and the curve have 0, 1 or 2 points of intersections? Show your working clearly. [3]

6 A polynomial, P, is $3x^3+5x^2-x+k$, where k is a constant.

(a) Find the value of k given that P leaves a remainder of 24 when divided by $x-2$. [2]

(b) In the case where $k=-3$, the quadratic expression $3x^2+ax^2-3$ is a factor of P. Find the value of the constant a. [4]

7 **(a)** Divide $2x^3 + 4x - 2$ by $x^3 + 2x$. [1]

(b) Express $\dfrac{2x^3 + 4x - 2}{x^3 + 2x}$ in partial fractions. [5]

(c) Hence, find $\int \dfrac{2x^3 + 4x^2 - 2}{x^3 + 2x}\, dx$. [2]

8 The line $3y + 4x = 12$ meets the x-axis at the point A and the y-axis at the point B.

(a) Write the coordinates of A and of B. [2]

The perpendicular bisector of AB meets the line $y = x$ at the point C.

(b) Find the coordinates of C. [4]

(c) Find the area of quadrilateral *OACB*, where *O* is the origin. [2]

9 **(a)** Write down the first three terms in the expansion of $\left(2-\frac{x}{4}\right)^n$, where n is a positive integer greater than 2, in ascending powers of x. [4]

The first two non-zero terms in the expansion of $(2+x)\left(2-\frac{x}{4}\right)^n$ in ascending powers of x are $a+bx^2$, where a and b are constants.

(b) Find the value of n. [2]

(c) Hence, find the value of a and of b. [2]

10 A curve is such that $\frac{d^2y}{dx^2} = 3e^{-x} + 8e^{2x}$. The curve intersects the *y*-axis at $P(0, -5)$ and has a gradient of 5 at *P*. Find the equation of the curve. [7]

11 (a) Show that $6\sin^2 x - 4\cos^2 x$ can be written as $a + b\cos 2x$, where a and b are integers. [2]

Hence,

(b) state the period and amplitude of $6\sin^2 x - 4\cos^2 x$, [2]

(c) Sketch the graph of $y = 6\sin^2 x - 4\cos^2 x$ for $0 \leq x \leq 2\pi$ radians. [3]

12 Liquid is poured, at a constant rate of $25\pi\ \text{cm}^3/\text{s}$, into a hemispherical bowl of radius r cm.

When the depth of the liquid directly below the centre of the bowl is x cm, the volume, V cm^3, of the liquid in the bowl is given by $V = \frac{1}{3}\pi x^2(3r - x)$.

It is given that the radius of hemispherical bowl of radius is 12 cm, find

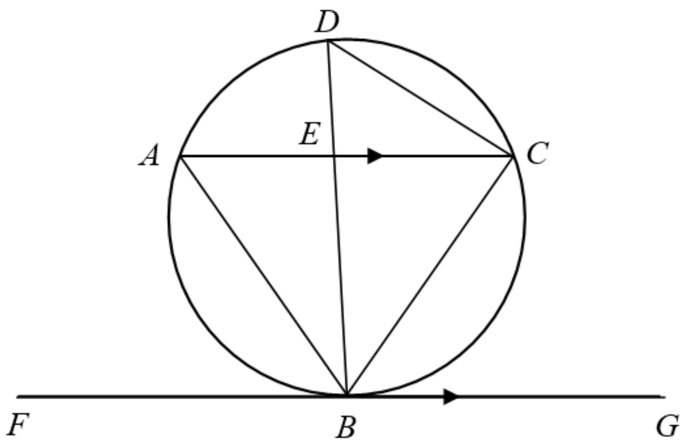
(a) the time taken for the depth of the liquid directly below the centre of the bowl to reach 6 cm, [3]

(b) the rate of change of the depth of liquid directly below the centre of the bowl at this time. [4]

13 **(a)** Show that $4\sec\theta + \tan\theta = 3\cot\theta$ can be expressed as $4\sin^2\theta + 4\sin\theta - 3 = 0$. [3]

(b) Hence, solve $4\sec 2x + \tan 2x = 3\cot 2x$ for $-180° < x < 180°$. [5]

14



The diagram shows triangles *ABC* and *BCD* whose vertices lie on the circumference of a circle. The chords *BD* and *AC* intersect at *E* and *AC* is parallel to *FG*. *FG* is a tangent to the circle at *B*.

Prove that

(a) ΔBCD is similar to ΔBEC, [3]

(b) $BC^2 = BD \times BE$, [2]

(c) ΔABC is an isosceles triangle. [2]

Answer Key

1	(a)	(2, 3)
	(b)	$k = 17$
2		$k = 4\frac{1}{2}$
3	(a)	$10^{11} : 1$
4		$a = -1$ and $b = 20$
5	(a)	$m = 6$ or $m = 2$
6	(a)	$k = -18$
	(b)	$a = 2$
7	(a)	$\frac{2x^3 + 4x - 2}{x^3 + 2x} = 2 - \frac{2}{x^3 + 2x}$
	(b)	$\frac{2x^3 + 4x - 2}{x^3 + 2x} = 2 - \frac{1}{x} + \frac{x}{x^2 + 2}$
	(c)	$2x - lnx + \frac{1}{2} ln(x^2 + 2) + c$ where c is a constant.
8	(a)	$A = (3, 0)$ and $B = (0, 4)$
	(b)	$C = \left(3\frac{1}{2}, 3\frac{1}{2}\right)$
	(c)	12.25 units^2
9	(a)	$2^n - n\left(2^{n-3}\right)x + n(n-1)2^{n-7}x^2 + ...$
	(b)	$n = 4$
	(c)	$a = 32$ and $b = -5$
10		$y = 3e^{-x} + 2e^{2x} + 4x - 10$
11	(a)	$1 - 5\cos 2x$
	(b)	Period = π Amplitude = 5
12	(a)	14.4 seconds
	(b)	$\frac{dx}{dt} = \frac{25}{108}$ cm/s
13	(b)	$x = -165°, -105°, \underline{15°, 75°}$

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[MARK SCHEME]

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(a) By expressing $4x^2 - 16x + 19$ in the form $a(x+b)^2 + c$, where a, b and c are constants, find the coordinates of the stationary point on the curve. [2]

$$\begin{aligned} 4x^2 - 16x + 19 &= 4(x^2 - 4x) + 19 \\ &= 4\left[(x-2)^2 - 4\right] + 19 \\ &= 4(x-2)^2 - 16 + 19 \\ &= 4(x-2)^2 + 3 \text{ [A1]} \end{aligned}$$

$\therefore$ Coordinates of the stationary point is (2, 3). [A1]

(b) The line $y = 4x + 3$ intersects the curve at the points A and B. Find the value of k for which the distance AB can be expressed as $3\sqrt{k}$. [4]

$y = 4x^2 - 16x + 19$ ....(1)

$y = 4x + 3$ ....(2)

Substitute (1) into (2):

$4x^2 - 16x + 19 = 4x + 3$

$4x^2 - 20x + 16 = 0$

$x^2 - 5x + 4 = 0$

$(x-1)(x-4) = 0$ [M1]

$x = 1$ or $x = 4$

$y = 7$ or $x = 19$

Coordinates of A and B are (1, 7) and (4, 19). [A1]

$$\begin{aligned} \text{Distance of } AB &= \sqrt{(4-1)^2 + (19-7)^2} \\ &= \sqrt{153} \text{ [A1]} \\ &= 3\sqrt{17} \end{aligned}$$

$\therefore k = 17$ [A1]

2 It is given that $\left(\sqrt{5}\right)^9-\left(\sqrt{5}\right)^7+\left(\sqrt{5}\right)^5-\left(\sqrt{5}\right)^3+105\left(\sqrt{5}\right)=5^k$. By factorisation, find the value of k. [4]

$$\left(\sqrt{5}\right)^9-\left(\sqrt{5}\right)^7+\left(\sqrt{5}\right)^5-\left(\sqrt{5}\right)^3+105\left(\sqrt{5}\right)=\sqrt{5}\left[\left(\sqrt{5}\right)^8-\left(\sqrt{5}\right)^6+\left(\sqrt{5}\right)^4-\left(\sqrt{5}\right)^2+105\right] \text{ [M1]}$$

$$=\sqrt{5}(5^4-5^3+5^2-5+105) \text{ [M1]}$$

$$=\sqrt{5}(625-125+25-5+105)$$

$$=\sqrt{5}(625)$$

$$=5^{\frac{1}{2}}\times 5^4$$

$$=5^{4\frac{1}{2}} \text{ [A1]}$$

$\therefore k=4\dfrac{1}{2}$ [A1]

3 The loudness of a sound can be measured using the equation $L = 10\lg\frac{I}{I_0}$, where I is the intensity of sound to be measured and I_0 is the intensity of sound that can barely be heard, also known as the threshold of hearing. The unit of L is the decibel (dB).

(a) Given that the loudness of a scream is 110 dB, find the ratio of the intensity of the scream to the threshold of hearing. [2]

$L = 10\lg\frac{I}{I_0}$

When $L = 110$,

$10\lg\frac{I}{I_0} = 110$

$\lg\frac{I}{I_0} = 11$

$\frac{I}{I_0} = 10^{11}$ [A1]

$\therefore$ Ratio is $10^{11}:1$ [A1]

(b) 130 dB is the pain threshold (the maximum level of sound we can hear without feeling intense pain and instantly damaging our hearing). [2]

Explain the impact, on hearing, the loudness of the sound of an unknown object falling from the sky onto Earth, if it has a sound intensity of $10^{-10.5}$ units and threshold of hearing of 10^{-25} units.

Given that $I = 10^{-10.5}$ and $I_0 = 10^{-25}$

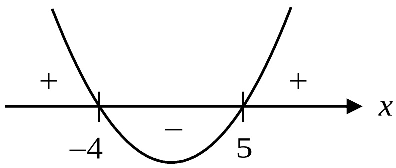
$L = 10\lg\frac{10^{-10.5}}{10^{-25}}$

$= 10\lg 10^{14.5}$

$= 145$ [A1]

Since the sound of the unknown object falling has a loudness of 145dB which <u>exceeds the pain threshold</u>, this can <u>cause damage</u> to our hearing. [A1]

4 Given that the range of values of x where $x^2 + ax < b$ is $-4 < x < 5$, find the value of a and of b. [4]



From the diagram,

$(x+4)(x-5) < 0$ [M1]

$x^2 - 5x + 4x - 20 < 0$

$x^2 - x < 20$ [A1]

Comparing $x^2 + ax < b$ with $x^2 - x < 20$, $a = -1$ [A1] and $b = 20$ [A1]

5 The line $y = mx + c$ is drawn on the same axes as the curve $y = 4x - 2x^2$.

(a) Given that the line is a tangent to the curve when $c = \frac{1}{2}$, find the possible values of m. [3]

$y = mx + \frac{1}{2}$................................(1)

$y = 4x - 2x^2$............................(2)

Substitute (1) into (2):

$mx + \frac{1}{2} = 4x - 2x^2$

$2x^2 - 4x + mx + \frac{1}{2} = 0$

$2x^2 + (m-4)x + \frac{1}{2} = 0$ [M1]

$a = 2, b = m - 4, c = \frac{1}{2}$

Since the line is a tangent to the curve,

Discriminant = 0

$(m-4)^2 - 4(2)\left(\frac{1}{2}\right) = 0$ [M1]

$(m-4)^2 = 4$

$m - 4 = 2$ or $m - 4 = -2$

$m = 6$ or $m = 2$ [A1]

(b) The line $y = mx + c$ has a negative y-intercept. Do the line and the curve have 0, 1 or 2 points of intersections? Show your working clearly. [3]

$y = mx + c$................................(3)

Substitute (3) into (2):

$mx + c = 4x - 2x^2$

$2x^2 - 4x + mx + c = 0$

$2x^2 + (m-4)x + c = 0$

Discriminant = $(m-4)^2 - 4(2)c$

= $(m-4)^2 - 8c$ [A1]

Since $(m-4)^2 \geq 0$ and $c < 0$, <u>discriminant > 0</u>. [A1]

$\therefore$ The line and the curve have 2 points of intersection. [A1]

6 A polynomial, P, is $3x^3 + 5x^2 - x + k$, where k is a constant.

(a) Find the value of k given that P leaves a remainder of 24 when divided by $x - 2$. [2]

Let $P = \mathrm{f}(x) = 3x^3 + 5x^2 - x + k$

Given that $\mathrm{f}(2) = 24$

$3(2)^3 + 5(2)^2 - 2 + k = 24$ [M1]

$24 + 20 - 2 + k = 24$

$k + 42 = 24$

$k = -18$ [A1]

(b) In the case where $k = -3$, the quadratic expression $3x^2 + ax^2 - 3$ is a factor of P. Find the value of the constant a. [4]

$3x^3 + 5x^2 - x - 3 = (3x^2 + ax - 3)(x + b)$ [M1]

$= 3x^3 + 3bx^2 + ax^2 + abx - 3x - 3b$

$= 3x^3 + (3b + a)x^2 + (ab - 3)x - 3b$ [M1]

Equating constants,

$-3b = -3$

$b = 1$ [A1]

Equating coefficients of x^2,

$3b + a = 5$

$3(1) + a = 5$

$a = 2$ [A1]

<u>Alternative Method</u>

Equating coefficients of x:

$ab - 3 = -1$

$a - 3 = -1$

$a = 2$

7 **(a)** Divide $2x^3+4x-2$ by x^3+2x. [1]

$$\begin{array}{r} 2 \\ x^3+2x \overline{\big) 2x^3+0x^2+4x-2} \\ \underline{-(2x^3 \qquad +4x)} \\ -2 \end{array}$$

$\therefore \dfrac{2x^3+4x-2}{x^3+2x}=2-\dfrac{2}{x^3+2x}$ [A1]

(b) Express $\dfrac{2x^3+4x-2}{x^3+2x}$ in partial fractions. [5]

$$\frac{2x^3+4x-2}{x^3+2x}=2+\frac{-2}{x(x^2+2)}$$

Let $\dfrac{-2}{x(x^2+2)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+2}$ [A1]

$-2=A(x^2+2)+x(Bx+C)$

Let $x=0$: $2A=-2$

$A=-1$ [A1]

Comparing coefficients of x^2: $-1+B=0$

$B=1$ [A1]

Comparing coefficients of $x: C=0$ [A1]

$\therefore \dfrac{2x^3+4x-2}{x^3+2x}=2-\dfrac{1}{x}+\dfrac{x}{x^2+2}$ [A1]

(c) Hence, find $\displaystyle\int \frac{2x^3+4x^2-2}{x^3+2x}\,dx$. [2]

$$\int \frac{2x^3+4x-2}{x^3+2x}\,dx=\int 2-\frac{1}{x}+\frac{x}{x^2+2}\,dx$$

$$=2x-lnx+\frac{1}{2}ln(x^2+2)+c$$ [A2: 1 mark per pair]

where c is a constant.

8 The line $3y+4x=12$ meets the x-axis at the point A and the y-axis at the point B.

(a) Write the coordinates of A and of B. [2]

When $y=0,\ 4x=12$

$x=3$

When $x=0,\ 3y=12$

$y=4$

$\therefore$ Coordinates of $A=(3,0)$ [B1] and $B=(0,4)$ [B1]

The perpendicular bisector of AB meets the line $y=x$ at the point C.

(b) Find the coordinates of C. [4]

Midpoint of $AB=\left(\frac{3+0}{2},\frac{0+4}{2}\right)$

$=\left(1\frac{1}{2},2\right)$ [A1]

Gradient of $AB=\frac{4-0}{0-3}$

$=-\frac{4}{3}$

Gradient of the perpendicular bisector of $AB=\frac{3}{4}$

Equation of the perpendicular bisector of AB:

$\frac{y-2}{x-1\frac{1}{2}}=\frac{3}{4}$ [M1]

$4y-8=3x-4\frac{1}{2}$

$4y=3x+3\frac{1}{2}$ ....(1) [A1]

Substitute $y=x$ into (1):

$4x=3x+3\frac{1}{2}$

$x=3\frac{1}{2}$

$\therefore$ Coordinates of $C=\left(3\frac{1}{2},3\frac{1}{2}\right)$ [A1]

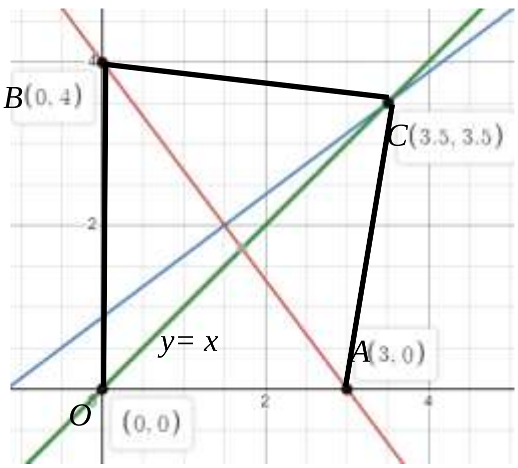
(c) Find the area of quadrilateral $OACB$, where O is the origin. [2]

$$\text{Area of } OACB = \frac{1}{2}\begin{vmatrix} 0 & 3 & 3\frac{1}{2} & 0 & 0 \\ 0 & 0 & 3\frac{1}{2} & 4 & 0 \end{vmatrix} \text{ [M1]}$$

$$= \frac{1}{2}\left\{\left[0 + 3\left(3\frac{1}{2}\right) + 3\frac{1}{2}(4) + 0\right] - 0\right\}$$

$$= \frac{1}{2}\left(\frac{49}{2}\right)$$

$$= 12\frac{1}{4} \text{ units}^2 \text{ [A1]}$$



<u>Alternative Method</u>

$$\text{Area of } OACB = \frac{1}{2}\times 4\times 3\frac{1}{2} + \frac{1}{2}\times 3\times 3\frac{1}{2} \text{ [M1]}$$

$$= 7 + 5\frac{1}{4}$$

$$= 12\frac{1}{4} \text{ units}^2 \text{ [A1]}$$

9 **(a)** Write down the first three terms in the expansion of $\left(2-\frac{x}{4}\right)^n$, where n is a positive integer greater than 2, in ascending powers of x. [4]

$$\left(2-\frac{x}{4}\right)^n = 2^n + \binom{n}{1}2^{n-1}\left(-\frac{x}{4}\right)^1 + \binom{n}{2}2^{n-2}\left(-\frac{x}{4}\right)^2 + \ldots \text{ [M1]}$$
$$= 2^n - n2^{n-1}\left(\frac{x}{2^2}\right) + \frac{n(n-1)}{2}2^{n-2}\left(\frac{x^2}{4^2}\right) + \ldots$$
$$= 2^n - n\left(2^{n-3}\right)x + n(n-1)2^{n-7}x^2 + \ldots \text{ [A3]}$$

The first two non-zero terms in the expansion of $(2+x)\left(2-\frac{x}{4}\right)^n$ in ascending powers of x are $a+bx^2$, where a and b are constants.

(b) Find the value of n. [2]

$$(2+x)\left(2-\frac{x}{4}\right)^n = (2+x)\left[2^n - n\left(2^{n-3}\right)x + n(n-1)2^{n-7}x^2 + \ldots\right]$$

Equating coefficients of x,

$$2\left[-n\left(2^{n-3}\right)\right] + 2^n = 0 \text{ [M1]}$$
$$2^n - n\left(2^{n-2}\right) = 0$$
$$2^n\left(1-\frac{n}{4}\right) = 0$$
$$2^n = 0 \text{ (N.A.) or } 1-\frac{n}{4} = 0$$
$$n = 4 \text{ [A1]}$$

(c) Hence, find the value of a and of b. [2]

Equating constants,

$$a = 2(2^4)$$
$$= 32 \text{ [B1]}$$

Equating coefficients of x^2,

$$b = 4(3)2^{-2} - 4(2^1)$$
$$= -5 \text{ [A1]}$$

10 A curve is such that $\frac{d^2y}{dx^2} = 3e^{-x} + 8e^{2x}$. The curve intersects the y-axis at $P(0,-5)$ and has a gradient of 5 at P. Find the equation of the curve. [7]

$\frac{d^2y}{dx^2} = 3e^{-x} + 8e^{2x}$

$\frac{dy}{dx} = -3e^{-x} + 4e^{2x} + c$ [M1]

When $x = 0$, $\frac{dy}{dx} = 5$,

$-3 + 4 + c = 5$ [M1]

$c = 4$ [A1]

$\frac{dy}{dx} = -3e^{-x} + 4e^{2x} + 4$ [A1]

$y = 3e^{-x} + 2e^{2x} + 4x + c$ [M1]

When $x = 0$, $y = -5$,

$3 + 2 + c = -5$

$c = -10$ [A1]

$\therefore$ Equation of the curve is $y = 3e^{-x} + 2e^{2x} + 4x - 10$ [A1]

11 **(a)** Show that $6\sin^2 x - 4\cos^2 x$ can be written as $a + b\cos 2x$, where *a* and *b* are integers. [2]

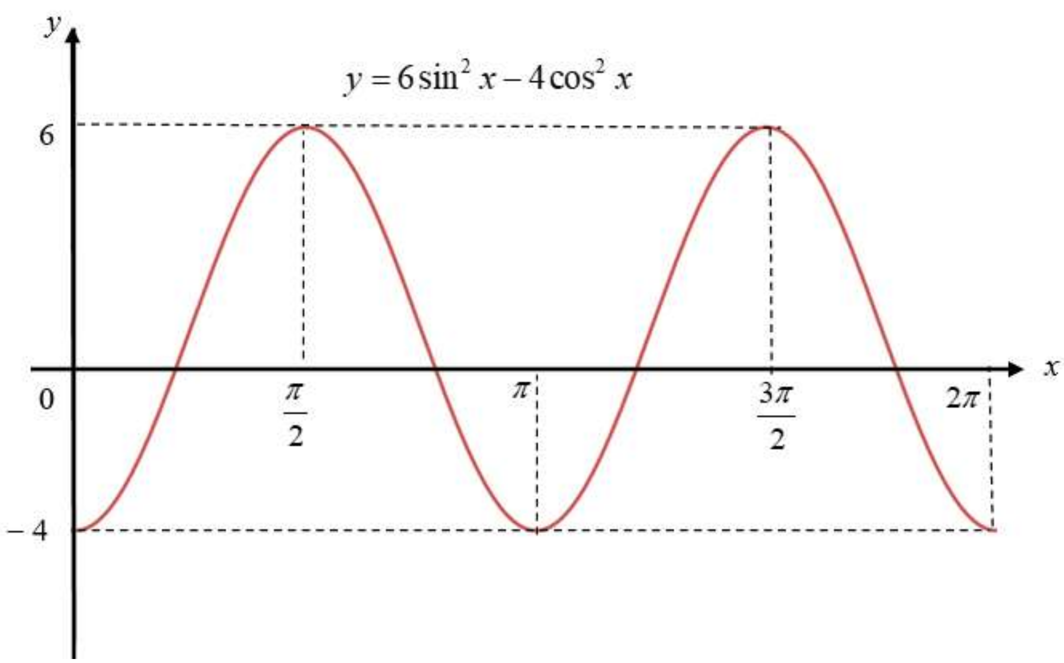
$$6\sin^2 x - 4\cos^2 x = 6\left(\frac{1-\cos 2x}{2}\right) - 4\left(\frac{1+\cos 2x}{2}\right) \text{ [M1]}$$
$$= 3 - 3\cos 2x - 2 - 2\cos 2x$$
$$= 1 - 5\cos 2x \text{ [A1]}$$

Hence,

(b) state the period and amplitude of $6\sin^2 x - 4\cos^2 x$, [2]

Period = π [B1] Accept 180°
Amplitude = 5 [B1]

(c) Sketch the graph of $y = 6\sin^2 x - 4\cos^2 x$ for $0 \le x \le 2\pi$ radians. [3]



B1: Correct shape from $0 \le x \le 2\pi$ with $(0,-4)$ and $(2\pi,-4)$
B1: Minimum points $(\pi,-4)$, $(0,-4)$ and $(2\pi,-4)$
B1: Maximum points $\left(\frac{\pi}{2},6\right)$ and $\left(\frac{3\pi}{2},6\right)$

12 Liquid is poured, at a constant rate of 25π cm^3/s, into a hemispherical bowl of radius r cm.

When the depth of the liquid directly below the centre of the bowl is x cm, the volume, V cm^3, of the liquid in the bowl is given by $V = \frac{1}{3}\pi x^2(3r - x)$.

It is given that the radius of hemispherical bowl of radius is 12 cm, find

(a) the time taken for the depth of the liquid directly below the centre of the bowl to reach 6 cm, [3]

$V = \frac{1}{3}\pi x^2(3r - x)$

Given that $r = 12$, $V = \frac{1}{3}\pi x^2(36 - x)$ [M1]

When $x = 6$, $V = \frac{1}{3}\pi(6)^2(36 - 6)$

$= 360\pi$ [A1]

$\therefore$ Time taken $= \frac{360\pi}{25\pi}$

$= 14.4$ seconds [A1]

(b) the rate of change of the depth of liquid directly below the centre of the bowl at this time. [4]

$V = \frac{1}{3}\pi x^2(36 - x)$

$= 12\pi x^2 - \frac{1}{3}\pi x^3$

$\frac{dv}{dx} = 24\pi x - \pi x^2$ [A1]

Given that $x = 6$ and $\frac{dv}{dt} = 25\pi$,

$\frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt}$

$25\pi = \left[24\pi(6) - \pi(6)^2\right] \times \frac{dx}{dt}$ [M1]

$= 108\pi \times \frac{dx}{dt}$ [M1]

$\frac{dx}{dt} = \frac{25}{108}$ cm/s [A1]

13 **(a)** Show that $4\sec\theta+\tan\theta=3\cot\theta$ can be expressed as $4\sin^2\theta+4\sin\theta-3=0$. [3]

$4\sec\theta+\tan\theta=3\cot\theta$

$\frac{4}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\frac{3\cos\theta}{\sin\theta}$ [M1]

$\frac{4+\sin\theta}{\cos\theta}=\frac{3\cos\theta}{\sin\theta}$

$4\sin\theta+\sin^2\theta=3\cos^2\theta$

$=3(1-\sin^2\theta)$ [M1]

$=3-3\sin^2\theta$

$4\sin^2\theta+4\sin\theta-3=0$ (shown) [A1]

(b) Hence, solve $4\sec 2x+\tan 2x=3\cot 2x$ for $-180°<x<180°$. [5]

$4\sec 2x+\tan 2x=3\cot 2x$

$4\sin^2 2x+4\sin 2x-3=0$ [A1]

$(2\sin 2x-1)(2\sin 2x+3)=0$ [M1]

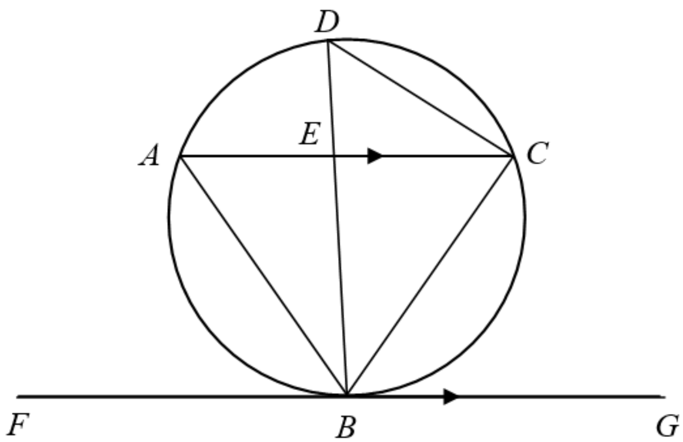
$\sin 2x=\frac{1}{2}$ or $\sin 2x=-1\frac{1}{2}$ (no solution) [M1]

Basic angle = 30°

$2x=-330°,-210°,30°,150°$

$x=-165°,-105°,\underline{15°,75°}$ [A1, <u>A1</u>]

14



The diagram shows triangles *ABC* and *BCD* whose vertices lie on the circumference of a circle. The chords *BD* and *AC* intersect at *E* and *AC* is parallel to *FG*. *FG* is a tangent to the circle at *B*.

Prove that

(a) ΔBCD is similar to ΔBEC, [3]

$\angle BDC = \angle CBG$ (alternate segment theorem)

$\angle BCE = \angle CBG$ (alternate angles, *AC*//*FG*)

$\therefore \angle BDC = \angle BCE$ [A1 with above 2 statements cited]

$\angle CBD = \angle EBC$ (common angle) [A1]

Since the <u>corresponding angles of the triangles are equal</u>, ΔBCD is similar to ΔBEC. [A1]

(b) $BC^2 = BD \times BE$, [2]

Since ΔBCD is similar to ΔBEC,

$\dfrac{BC}{BD} = \dfrac{BE}{BC}$ [M1]

$BC^2 = BD \times BE$ [A1]

(c) ΔABC is an isosceles triangle. [2]

$\angle BDC = \angle BCE$ (from (a))
$\angle BDC = \angle BAC$ ($\angle$ in same segment)
$\angle BCE = \angle BAC$ [A1 with above cited]
$\therefore \Delta ABC$ is an isosceles triangle. [A1]

Alternative method,
$C\hat{B}G = A\hat{C}B$ (alternate angles, $AC // FG$)
$C\hat{B}G = B\hat{A}C$ (angles in alternate segment)
$A\hat{C}B = B\hat{A}C$ [A1 with above cited]
$\therefore \Delta ABC$ is an isosceles triangle. [A1]

HILLGROVE SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2023
SECONDARY FOUR (EXPRESS)

CANDIDATE NAME	( )	CLASS	-

CENTRE NUMBER	S					INDEX NUMBER				

Additional Mathematics **4049/02**

Paper 2 **29 August 2023**

Candidates answer on the Question Paper. **2 hours 15 minutes**

No Additional Materials are required. **10.05 a.m. – 12.20 p.m.**

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You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
TOTAL	**90**

Parent's/ Guardian's Signature: ____________________

Setters: Mdm Lee Li Lian

This document consists of **24** printed pages, including this page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Binomial expansion

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{r}a^{n-r}b^r+\ldots+b^n,$$

where n is a positive integer and $\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A+\cos^2 A=1$$

$$\sec^2 A=1+\tan^2 A$$

$$\operatorname{cosec}^2 A=1+\cot^2 A$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\sin 2A=2\sin A\cos A$$

$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$a^2=b^2+c^2-2bc\cos A$$

$$\Delta=\frac{1}{2}ab\sin C$$

1 A calculator must not be used in this question.

(a) Show that $\tan 15° = 2 - \sqrt{3}$. [4]

(b) Use the result from **part (a)** to find an expression for $\sec^2 15°$, in the form $a + b\sqrt{3}$ where a and b are integers. [2]

2 **(a)** Given that $\int_{-5}^{2} \mathrm{f}(x)dx = \int_{2}^{3} \mathrm{f}(x)dx = 5$, find $\int_{-5}^{3} 3\left[\mathrm{f}(x) - x\right]dx$. [4]

(b) Differentiate $5x^2 \ln x$ with respect to x. Hence, find the value of $\int_1^3 5x \ln x \, dx$, giving your answer correct to 2 decimal places. [4]

3 **(a)** Solve the equation $5^x - 25^{x-1} - 6 = 0$. [5]

(b) **(i)** Given that $\log_{343} x^3 = \log_{49} y$, express y in terms of x. [3]

(ii) Find the value of x for which $\log_{49}(x^2 + 11x) - \log_{343} x^3 = \dfrac{1}{\log_{49} 7}$. [3]

4 A particle travels in a straight line so that, t seconds after leaving fixed point, O, its velocity is, v ms^{-1}, is given by $v = t^2 - 8kt + 6k$, where k is a constant. The minimum velocity of the particle occurs when $t = 12$.

(a) Show that $k = 3$. [2]

(b) Determine whether the particle will return to O during its journey. [4]

(c) Find the total distance travelled by the particle in the first 2 seconds. [3]

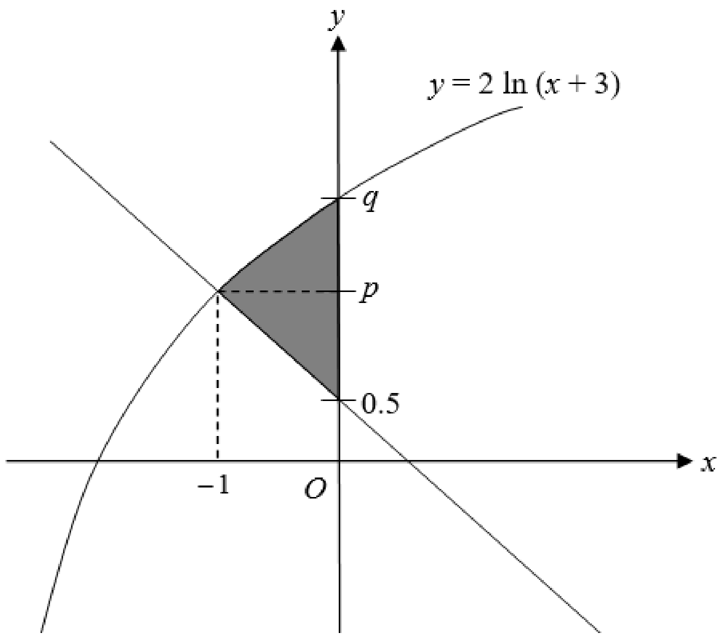
5 It is given that $\mathrm{f}(x) = 11 - ax - x^2 = 36 - (b + x)^2$, where *a* and *b* are both positive, for all real values of *x*.

(a) Find the value of *a* and of *b*. [2]

(b) Determine if $\mathrm{f}(x)$ has a maximum or minimum value, state this value. [2]

(c) Find the range of values of *x* for which $\mathrm{f}(x)$ is positive. [3]

6 In the diagram, the curve $y = 2\ln(x + 3)$ cuts the y-axis at $(0, q)$. A line, which meets the curve at $(-1, p)$ cuts the y-axis at $(0, 0.5)$.



(a) State the exact value of p and of q. [2]

(b) Calculate the area of the shaded region. [4]

7 **(a)** A formula for working out the braking distance, d for a vehicle travelling at a speed v, is $d = av^3 + bv^2$, where a and b are constants. Values of d for different values of v have been collected. [4]

Explain how a straight line can be drawn to represent the formula, and state how the values of a and b could be obtained from the line.

(b) The value, \$$V$, of an art piece has been increasing each year from 2008 to 2020. An auctioneer claims that the increase is exponential and so can be modelled by an equation in the form

$$V = V_o e^{kt},$$

where V_o and k are constants and t is the time in years since 1st January 2008. The table below gives values of V and t for some of the years from 2008 to 2017.

Year	2008	2011	2014	2017
t years	0	3	6	9
\$$V$	12000	12900	13900	15000

(i) Plot $\ln V$ against t and draw a straight line graph to show that the model is valid for the years 2008 to 2020. [2]

(ii) Estimate the value of V_o and k. [3]

(iii) Explain the significance of the value of V_o. [1]

(iv) Assuming that the model is still appropriate, estimate the value of the art piece on 1^{st} January 2020. [2]

8 **(a)** Show that $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ where x is measured in radians. [2]

(b) A musician wants to superimpose two sound waves to form an overall sound. Two such sound waves are $\mathrm{f}(t)$ and $\mathrm{g}(t)$ where, for $t \geq 0$ (in seconds),

$\mathrm{f}(t)=12\sin\left(\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$ and $\mathrm{g}(t)=3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)-4\sin\left(\frac{t}{4}\right)$.

The overall sound $\mathrm{C}(t)$ is found by adding the two sound waves $\mathrm{f}(t)$ and $\mathrm{g}(t)$.

Using the result from (a),

(i) show that the overall sound wave $\mathrm{C}(t)$ may be written in the form [2]

$$\mathrm{C}(t)=a\sin\left(\frac{t}{4}\right)+b\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$$

where a and b are integers to be determined.

$C(t)$ may also be written in the form $C(t) = R\sin\left(\frac{t}{4}+\alpha\right)$, where R is a positive constant and α is an acute angle measured in radians.

(ii) Find the value of $\tan\alpha$ and R. [4]

(iii) Find the time, in seconds, at which the overall sound wave is first at its minimum. [3]

9 Three points are given by $P(3,-3)$, $Q(11,1)$ and $R(9,5)$.

(a) Show that angle PQR is $90°$. [3]

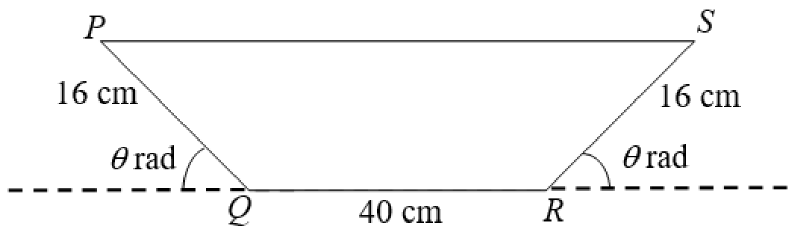
(b) Explain why P, Q and R lie on a circle with diameter PR. [1]

(c) Find the equation of the circle in general form. [3]

(d) Explain why the tangent to the circle at Q is parallel to the y-axis. [2]

(e) Find the equation of the tangent to the circle at *R*. [2]

10 The diagram shows the vertical cross-section $PQRS$ of an open trough made from plastic sheeting. The lengths of PQ, QR and RS are 16 cm, 40 cm and 16 cm respectively. The trough rests with QR on horizontal ground and both PQ and RS are inclined at θ radians to the ground.



(a) Show that the area, A cm^2, of the cross-section $PQRS$ is given by [4]

$$A = 640\sin\theta + 128\sin 2\theta\,.$$

(b) Given that θ can vary, find the value of θ for which the trough can hold a maximum amount of water. [5]

Answer Key

1	(a)		$2-\sqrt{3}$
	(b)		$8-4\sqrt{3}$
2	(a)		54
	(b)		14.72
3	(a)		1.43 or 1.68
	(b)	(i)	$y = x^2$
		(ii)	$x = 0$ (N.A.) or $x = \frac{11}{2400}$
4	(c)		23.1 m
5	(a)		a = 10
	(b)		The maximum value of $f(x) = 36$
	(c)		$-11 < x < 1$
6	(a)		$p = 2\ln 2$ $q = 2\ln 3$
	(b)		0.876 units^2
7	(a)		Plot $\frac{d}{v^2}$ against v. Gradient $= a$ Vertical-intercept $= b$
	(b)	(ii)	$V_o \approx 12000$ (to 3 s.f.) Gradient, $k = 0.025$ [Accept $\pm 0.0025 =$ 0.0225 to 0.0275]
		(iv)	$16155.24 [Accept $15677.78 to $16647.24]
8	(b)	(i)	$a = 8$ and $b = 6$
		(ii)	$\tan\alpha = \frac{3}{4}$ and $R = 10$
		(iii)	$t \approx 16.3$
9	(c)		$x^2 - 12x + y^2 - 2y + 12 = 0$
	(e)		$4y = -3x + 47$
10	(b)		$\theta \approx 1.25$ radians

HILLGROVE SECONDARY SCHOOL
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[MARK SCHEME]

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$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

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Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\Delta = \frac{1}{2}ab\sin C$$

1 A calculator must not be used in this question.

(a) Show that $\tan 15° = 2-\sqrt{3}$. [4]

$$\tan 15° = \tan(45°-30°)$$

$$= \frac{\tan 45° - \tan 30°}{1+\tan 45° \tan 30°}$$

$$= \frac{1-\frac{\sqrt{3}}{3}}{1+1\times\frac{\sqrt{3}}{3}} \text{ [A1]}$$

$$= \frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}$$

$$= \frac{3-\sqrt{3}}{3} \div \frac{3+\sqrt{3}}{3}$$

$$= \frac{3-\sqrt{3}}{3} \times \frac{3}{3+\sqrt{3}}$$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} \times \frac{3-\sqrt{3}}{3-\sqrt{3}} \text{ [M1]}$$

$$= \frac{9-6\sqrt{3}+3}{9-3} \text{ [M1]}$$

$$= \frac{12-6\sqrt{3}}{6}$$

$$= 2-\sqrt{3} \text{ [A1]}$$

<u>Alternative Method</u>

$$\tan 15° = \tan(60°-45°)$$

$$= \frac{\tan 60° - \tan 45°}{1+\tan 60° \tan 45°}$$

$$= \frac{\sqrt{3}-1}{1+\sqrt{3}\times 1} \text{ [A1]}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \text{ [M1]}$$

$$= \frac{3-2\sqrt{3}+1}{2} \text{ [M1]}$$

$$= 2-\sqrt{3} \text{ [A1]}$$

(b) Use the result from **part (a)** to find an expression for $\sec^2 15°$, in the form $a + b\sqrt{3}$ where a and b are integers. [2]

$$\begin{aligned} \sec^2 15° &= 1 + \tan^2 15° \\ &= 1 + \left(2 - \sqrt{3}\right)^2 \\ &= 1 + \underline{4 - 4\sqrt{3} + 3} \text{ [A1]} \\ &= 8 - 4\sqrt{3} \text{ [A1]} \end{aligned}$$

2 **(a)** Given that $\int_{-5}^{2} \mathrm{f}(x)dx = \int_{2}^{3} \mathrm{f}(x)dx = 5$, find $\int_{-5}^{3} 3\left[\mathrm{f}(x) - x\right]dx$. [4]

$\int_{-5}^{3} 3\left[\mathrm{f}(x) - x\right]dx$

$= 3\int_{-5}^{3} \mathrm{f}(x)\ dx - 3\int_{-5}^{3} x\ dx$

$= 3\left[\underline{\int_{-5}^{2} \mathrm{f}(x)\ dx + \int_{2}^{3} \mathrm{f}(x)\ dx}\right] - 3\int_{-5}^{3} x\ dx$ [M1]

$= 3(5+5) - 3\left[\dfrac{x^2}{2}\right]_{-5}^{3}$ [M1]

$= 30 - \dfrac{3}{2}(9-25)$ [M1]

$= 54$ [A1]

(b) Differentiate $5x^2 \ln x$ with respect to x. Hence, find the value of $\int_1^3 5x \ln x \, dx$, giving your answer correct to 2 decimal places. [4]

$$\frac{d}{dx}(5x^2 \ln x) = 5x^2 \times \frac{1}{x} + \ln x \times 10x$$

$$= 5x + 10x \ln x \text{ [A1]}$$

$$\int_1^3 (5x + 10x \ln x)\, dx = \left[5x^2 \ln x\right]_1^3 \text{ [M1]}$$

$$\int_1^3 5x\, dx + \int_1^3 10x \ln x\, dx = \left[5x^2 \ln x\right]_1^3$$

$$\int_1^3 10x \ln x\, dx = \left[5x^2 \ln x\right]_1^3 - \int_1^3 5x\, dx$$

$$\frac{1}{2}\int_1^3 10x \ln x\, dx = \frac{1}{2}\left[5x^2 \ln x\right]_1^3 - \frac{1}{2}\int_1^3 5x\, dx$$

$$\therefore \int_1^3 5x \ln x\, dx = \frac{1}{2}\left(45\ln 3 - 5\ln 1\right) - \underline{\frac{1}{2}\left[\frac{5x^2}{2}\right]_1^3} \text{ [A1]}$$

$$= \frac{1}{2}\left(45\ln 3 - 5\ln 1\right) - \frac{1}{2}\left(\frac{45}{2} - \frac{5}{2}\right)$$

$$= \frac{1}{2}\left(45\ln 3 - 5\ln 1\right) - \frac{1}{2}\left(\frac{45}{2} - \frac{5}{2}\right)$$

$$= 14.7187765$$

$$\approx 14.72 \text{ (to 2 d.p.) [A1]}$$

3 **(a)** Solve the equation $5^x - 25^{x-1} - 6 = 0$. [5]

$5^x - 25^{x-1} - 6 = 0$

$5^x - 5^{2x-2} - 6 = 0$

$5^x - \dfrac{(5^x)^2}{25} - 6 = 0$ [M1]

$25 \times 5^x - (5^x)^2 - 150 = 0$

Let $y = 5^x$,

$25y - y^2 - 150 = 0$

$y^2 - 25y + 150 = 0$

$(y-10)(y-15) = 0$ [M1]

$y = 10$	or	$y = 15$
$5^x = 10$		$5^x = 15$ [A1]
$\lg 5^x = \lg 10$		$\lg 5^x = \lg 15$
$x \lg 5 = \lg 10$		$x \lg 5 = \lg 15$
$x = \dfrac{\lg 10}{\lg 5}$		$x = \dfrac{\lg 15}{\lg 5}$ [M1 for either shown]
$= 1.430676558$		$= 1.682606194$
≈ 1.43 (to 3 s.f.)		≈ 1.68 (to 3 s.f.) [A1 for both]

(b) **(i)** Given that $\log_{343} x^3 = \log_{49} y$, express y in terms of x. [3]

$$\log_{343} x^3 = \log_{49} y$$

$$\frac{\log_7 x^3}{\log_7 343} = \frac{\log_7 y}{\log_7 49} \text{ [M1]}$$

$$\frac{3\log_7 x}{3\log_7 7} = \frac{\log_7 y}{2\log_7 7}$$

$$\log_7 x = \frac{1}{2}\log_7 y \text{ [A1]}$$

$$= \log_7 \sqrt{y}$$

$$x = \sqrt{y}$$

$$y = x^2 \text{ [A1]}$$

Alternative Method

$$\log_7 x = \frac{1}{2}\log_7 y \text{ [A1]}$$

$$2\log_7 x = \log_7 y$$

$$y = x^2 \text{ [A1]}$$

(ii) Find the value of x for which $\log_{49}(x^2 + 11x) - \log_{343} x^3 = \dfrac{1}{\log_{49} 7}$. [3]

$$\log_{49}(x^2+11x) - \log_{343} x^3 = \frac{1}{\log_{49} 7}$$

$$\log_{49}(x^2+11x) - \log_{49} x^2 = \frac{1}{\log_{49} 7} \text{ [M1]}$$

$$\log_{49}(x^2+11x) - \log_{49} x^2 = \frac{1}{\frac{\log_7 7}{\log_7 49}}$$

$$\log_{49} \frac{x^2+11x}{x^2} = 2 \text{ [M1]}$$

$$\frac{x^2+11x}{x^2} = 49^2$$

$$\frac{x^2+11x}{x^2} = 2401$$

$$2401x^2 = x^2 + 11x$$

$$2400x^2 - 11x = 0$$

$$x(2400x - 11) = 0$$

$$x = 0 \text{ (N.A.) or } x = \frac{11}{2400} \text{ [A1]}$$

4 A particle travels in a straight line so that, t seconds after leaving fixed point, O, its velocity is, v ms^{-1}, is given by $v = t^2 - 8kt + 6k$, where k is a constant. The minimum velocity of the particle occurs when $t = 12$.

(a) Show that $k = 3$. [2]

$v = t^2 - 8kt + 6k$

Acceeration, $a = \frac{dv}{dt}$

$= 2t - 8k$ [A1]

When $t = 12$, $\frac{dv}{dt} = 0$

$24 - 8k = 0$

$k = 3$ (shown) [A1]

(b) Determine whether the particle will return to O during its journey. [4]

$v = t^2 - 24t + 18$

Displacement, $s = \frac{t^3}{3} - \frac{24t^2}{2} + 18t + c$, where c is a constant

When $t = 0, s = 0$,

$c = 0$

$\therefore s = \frac{t^3}{3} - 12t^2 + 18t$ [A1]

When $s = 0$,

$\frac{t^3}{3} - 12t^2 + 18t = 0$ [M1]

$t^3 - 36t^2 + 54t = 0$

$t(t^2 - 36t + 54) = 0$

$t = 0$ or $t^2 - 36t + 54 = 0$

$$t = \frac{-(-36) \pm \sqrt{(-36)^2 - 4(1)(54)}}{2(1)}$$

$$= \frac{36 \pm \sqrt{1080}}{2}$$

$= 34.43167673$ or 1.568323275 [A1]

$\therefore$ Yes, the particle will return to O at $t \approx 1.57$s and 34.4s. [A1]

(c) Find the total distance travelled by the particle in the first 2 seconds. [3]

When $v = 0$,

$t^2 - 24t + 18 = 0$

$t = \dfrac{24 \pm \sqrt{24^2 - 4\times1\times18}}{2\times1}$

$= \dfrac{24 \pm \sqrt{504}}{2}$

$= 23.22497216$ or 0.7750278397

≈ 23.2 or 0.775 [M1]

$s = \dfrac{t^3}{3} - 12t^2 + 18t$

When $t = 0, s = 0$ m

When $\underline{t = 0.7750278397, s = 6.897661467 \text{ m}}$ [<u>Either this or below M1</u>]

When $\underline{t = 2, s = -9\frac{1}{3} \text{ m}}$

Total distance travelled by the particle in the first 2 seconds

$= 2\times 6.897661467 + 9\frac{1}{3}$

$= 23.12865627$

≈ 23.1 m [A1]

5 It is given that $f(x) = 11 - ax - x^2 = 36 - (b+x)^2$, where *a* and *b* are both positive, for all real values of *x*.

(a) Find the value of *a* and of *b*. [2]

$f(x) = 11 - ax - x^2 = 36 - (b+x)^2$

$f(0) = 11 = 36 - b^2$

$b^2 = 25$

$b = 5$ [A1] or $b = -5$ (N.A., $\because b$ is positive)

$f(1) = 11 - a - 1 = 36 - (5+1)^2$

$10 - a = 0$

$a = 10$ [A1]

(b) Determine if $f(x)$ has a maximum or minimum value, state this value. [2]

$f(x) = 11 - 10x - x^2 = 36 - (5+x)^2$

The maximum [A1] value of $f(x) = 36$ [A1]

(c) Find the range of values of *x* for which $f(x)$ is positive. [3]

When $f(x) > 0$,

$36 - (5+x)^2 > 0$

$36 - (25 + 10x + x^2) > 0$

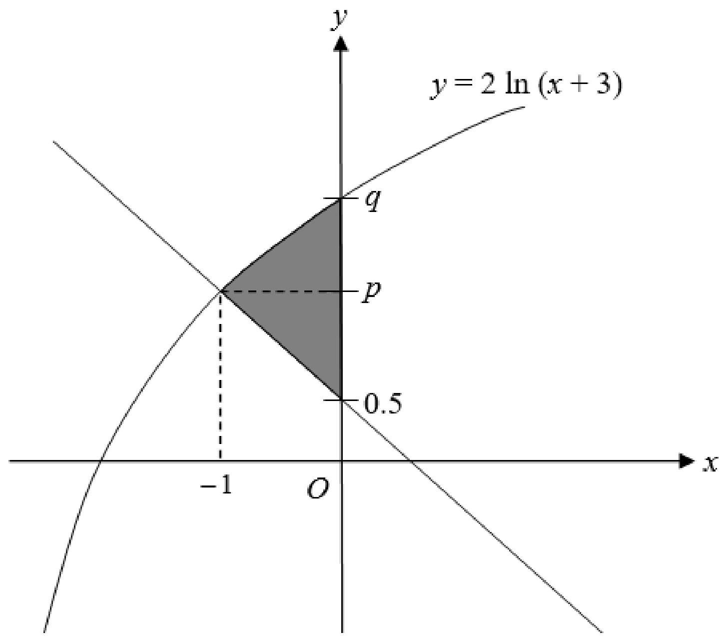
$36 - 25 - 10x - x^2 > 0$

$x^2 + 10x - 11 < 0$ [M1]

$(x-1)(x+11) < 0$ [M1]

$-11 < x < 1$ [A1]

6 In the diagram, the curve $y = 2\ln(x+3)$ cuts the y-axis at $(0, q)$. A line, which meets the curve at $(-1, p)$ cuts the y-axis at $(0, 0.5)$.



(a) State the exact value of p and of q. [2]

$p = 2\ln 2$ [Accept: $\ln 4$] [B1]
$q = 2\ln 3$ [Accept: $\ln 9$] [B1]

(b) Calculate the area of the shaded region. [4]

$$\text{Area of } \Delta = \frac{1}{2}(1)(\ln 4 - 0.5)$$
$$= 0.4431471806 \text{ [A1]}$$

$$y = 2\ln(x+3)$$
$$e^{\frac{y}{2}} = x+3$$
$$x = e^{\frac{y}{2}} - 3$$

Area of shaded region

$$= 0.4431471806 + \left| \int_{\ln 4}^{\ln 9} e^{\frac{y}{2}} - 3 \, \mathrm{d}y \right| \text{ [M1]}$$
$$= 0.4431471806 + \left| \left[2e^{\frac{y}{2}} - 3y \right]_{\ln 4}^{\ln 9} \right| \text{ [M1]}$$
$$= 0.4431471806 + \left| \left(2e^{\frac{\ln 9}{2}} - 3\ln 9 \right) - \left(2e^{\frac{\ln 4}{2}} - 3\ln 4 \right) \right|$$
$$= 0.4431471806 + 0.4327906486$$
$$= 0.8759378292$$
$$\approx 0.876 \text{ units}^2 \text{ (to 3 s.f.) [A1]}$$

7 **(a)** A formula for working out the braking distance, d for a vehicle travelling at a speed v, is $d = av^3 + bv^2$, where a and b are constants. Values of d for different values of v have been collected. [4]

Explain how a straight line can be drawn to represent the formula, and state how the values of a and b could be obtained from the line.

$d = av^3 + bv^2$

$\frac{d}{v^2} = av + b$ [A1]

Plot $\frac{d}{v^2}$ against v. [B1]

Gradient $= a$ [B1]

Vertical-intercept $= b$ [B1]

<u>Alternative Method</u>

$d = av^3 + bv^2$

$\frac{d}{v^3} = a + b\left(\frac{1}{v}\right)$ [A1]

Plot $\frac{d}{v^3}$ against $\frac{1}{v}$. [B1]

Gradient $= b$ [B1]

Vertical-intercept $= a$ [B1]

(b) The value, $\$V$, of an art piece has been increasing each year from 2008 to 2020. An auctioneer claims that the increase is exponential and so can be modelled by an equation in the form

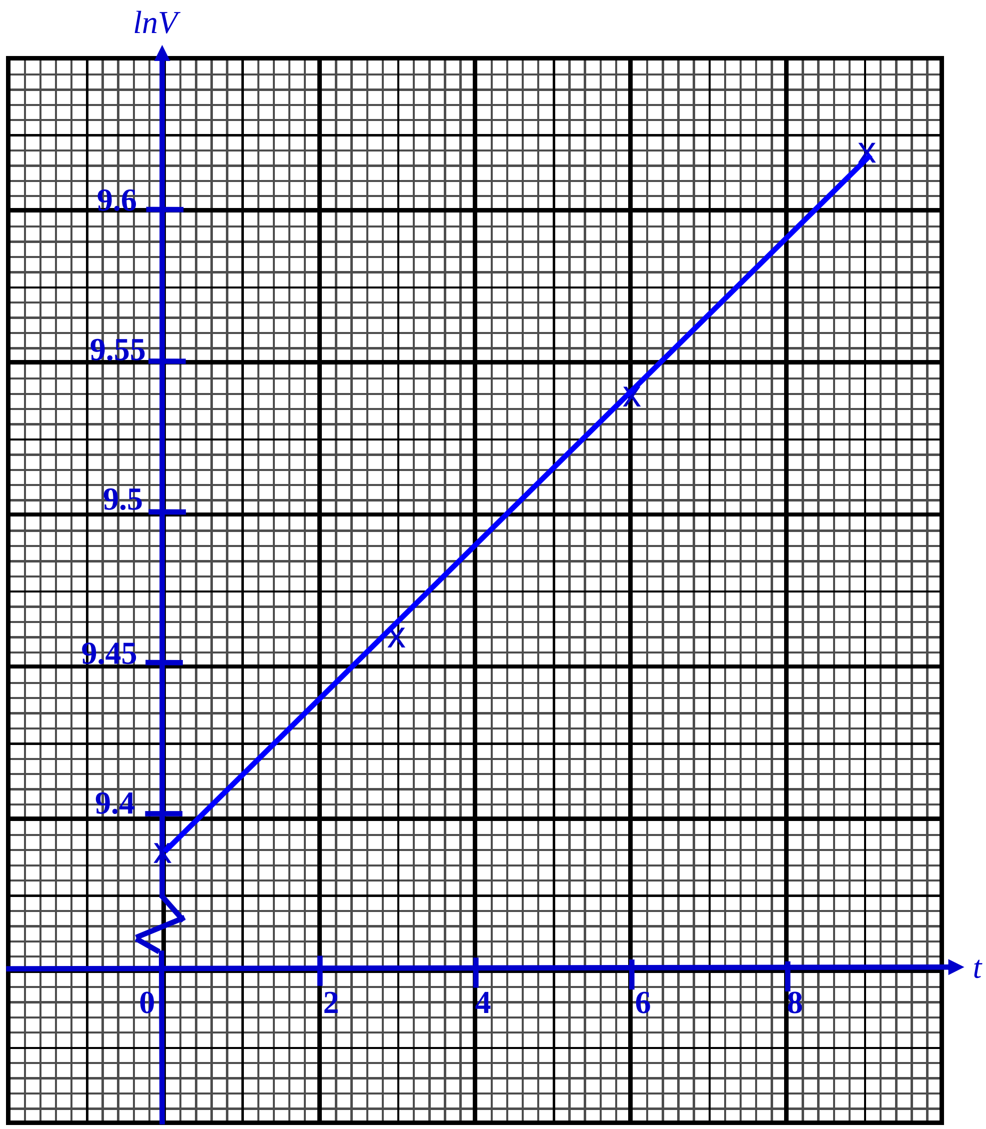
$$V = V_o e^{kt},$$

where V_o and k are constants and t is the time in years since 1st January 2008. The table below gives values of V and t for some of the years from 2008 to 2017.

Year	2008	2011	2014	2017
t years	0	3	6	9
$\$V$	12000	12900	13900	15000

(i) Plot $\ln V$ against t and draw a straight line graph to show that the model is valid for the years 2008 to 2020. [2]

t years	0	3	6	9
$\ln V$ [B1]	9.39	9.46	9.54	9.62



Best-fit line with vertical-axis intercept [B1]

(ii) Estimate the value of V_o and k. [3]

$$V = V_o e^{kt}$$
$$\ln V = \ln V_o e^{kt}$$
$$= \ln V_o + \ln e^{kt}$$
$$= \ln V_o + kt \ln e$$
$$= kt + \ln V_o$$

Vertical-axis intercept = $InV_o = 9.39$

$$V_o = e^{9.39}$$
$$= 11968.09933$$
$$\approx 12000 \text{ (to 3 s.f.) [A1]}$$

Gradient, $k = \dfrac{9.59 - 9.41}{8 - 0.8}$ [M1]

$= 0.025$ [A1] [Accept $\pm 0.0025 =$ 0.0225 to 0.0275]

(iii) Explain the significance of the value of V_o. [1]

It refers to the <u>value of the art piece on 1st January 2008</u>. (Accept initial value of the art piece. [B1]

(iv) Assuming that the model is still appropriate, estimate the value of the art piece on 1st January 2020. [2]

$$V = 11968.09933e^{0.02465277778t}$$

When $t = 12$, $V = 11968.09933e^{0.025(12)}$ [M1]

$= 16155.24429$

$\therefore$ The value of the art piece was about \$16155.24 [A1]

Accept \$15677.78 to \$16647.24

8 **(a)** Show that $\sin\left(\frac{\pi}{2}-x\right)=\cos x$ where x is measured in radians. [2]

$$\sin\left(\frac{\pi}{2}-x\right)=\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x \text{ [M1]}$$
$$=1\times\cos x-0\times\sin x$$
$$=\cos x \text{ (shown) [A1]}$$

(b) A musician wants to superimpose two sound waves to form an overall sound. Two such sound waves are $\mathrm{f}(t)$ and $\mathrm{g}(t)$ where, for $t\geq 0$ (in seconds),

$\mathrm{f}(t)=12\sin\left(\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$ and $\mathrm{g}(t)=3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)-4\sin\left(\frac{t}{4}\right)$.

The overall sound $\mathrm{C}(t)$ is found by adding the two sound waves $\mathrm{f}(t)$ and $\mathrm{g}(t)$.

Using the result from (a),

(i) show that the overall sound wave $\mathrm{C}(t)$ may be written in the form [2]

$$\mathrm{C}(t)=a\sin\left(\frac{t}{4}\right)+b\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$$

where a and b are integers to be determined.

$$\mathrm{C}(t)=12\sin\left(\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)+3\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)-4\sin\left(\frac{t}{4}\right)$$
$$=8\sin\left(\frac{t}{4}\right)+6\sin\left(\frac{\pi}{2}-\frac{t}{4}\right) \text{ [A1]}$$

$\therefore a=8$ and $b=6$ [A1]

$C(t)$ may also be written in the form $C(t) = R\sin\left(\frac{t}{4}+\alpha\right)$, where R is a positive constant and α is an acute angle measured in radians.

(ii) Find the value of $\tan\alpha$ and R. [4]

$$C(t) = 8\sin\left(\frac{t}{4}\right) + 6\sin\left(\frac{\pi}{2}-\frac{t}{4}\right)$$

$$= 8\sin\left(\frac{t}{4}\right) + 6\cos\left(\frac{t}{4}\right) \text{ [A1]}$$

$$8\sin\left(\frac{t}{4}\right) + 6\cos\left(\frac{t}{4}\right) = R\sin\left(\frac{t}{4}+\alpha\right)$$

$$= R\sin\frac{t}{4}\cos\alpha + R\cos\frac{t}{4}\sin\alpha \text{ [M1]}$$

$R\cos\alpha = 8$....................................(1)

$R\sin\alpha = 6$....................................(2)

$$\frac{(2)}{(1)} : \tan\alpha = \frac{6}{8}$$

$$= \frac{3}{4} \text{ [A1]}$$

$$R = \sqrt{8^2+6^2} = 10 \text{ [A1]}$$

(iii) Find the time, in seconds, at which the overall sound wave is first at its minimum. [3]

$\tan\alpha = \frac{3}{4}$

$\alpha = 0.6435011088$ radians [A1]

$C(t) = 10\sin\left(\frac{t}{4} + 0.6435011088\right)$

When $C(t) = -10$

$10\sin\left(\frac{t}{4} + 0.6435011088\right) = -10$ [M1]

$\sin\left(\frac{t}{4} + 0.6435011088\right) = -1$

$\frac{t}{4} + 0.6435011088 = \frac{3\pi}{2}$

$\frac{t}{4} = 4.068887872$

$t = 16.27555149$

≈ 16.3 (to 3 s.f.) [A1]

9 Three points are given by $P(3,-3)$, $Q(11,1)$ and $R(9,5)$.

(a) Show that angle *PQR* is 90°. [3]

$$\text{Gradient of } PQ, m_{PQ} = \frac{-3-1}{3-11}$$
$$= \frac{-4}{-8}$$
$$= \frac{1}{2} \text{ [A1]}$$
$$\text{Gradient of } QR, m_{QR} = \frac{5-1}{9-11}$$
$$= \frac{4}{-2}$$
$$= -2 \text{ [A1]}$$

Since $m_{PQ} \times m_{QR} = -1$, then PQ is perpendicular to QR and

$P\hat{Q}R = 90°$ (shown) [A1]

<u>Alternative Method</u>

Length of $PQ = \sqrt{(11-3)^2 + (1-(-3))^2}$ [M1]

$= \sqrt{80}$ units

Length of $QR = \sqrt{(11-9)^2 + (1-5)^2}$

$= \sqrt{20}$ units

Length of $PR = \sqrt{(3-9)^2 + (-3-5)^2}$

$= 10$ units

Since $\underline{PQ^2 + QR^2 = PR^2}$ [A1] ,∴ by converse of <u>Pythagoras' Theorem</u>, [A1]

PQ is perpendicular to QR and $P\hat{Q}R = 90°$ (shown)

(b) Explain why *P*, *Q* and *R* lie on a circle with diameter *PR*. [1]

By converse of <u>right angle in a semicircle</u>, since $P\hat{Q}R = 90°$, then *P*, *Q* and *R* lie on a circle with diameter *PR*.

(c) Find the equation of the circle in general form. [3]

$PR = \sqrt{(3-9)^2 + (-3-5)^2}$

$= 10$ units

Radius $= \frac{1}{2} \times 10$

$= 5$ units [A1]

Midpoint of $PR = \left(\frac{3+9}{2}, \frac{-3+5}{2} \right)$

= (6, 1)

Centre of the circle = (6, 1) [A1]

Equation of the circle:

$(x-6)^2 + (y-1)^2 = 5^2$

$x^2 - 12x + 36 + y^2 - 2y + 1 = 25$

$x^2 - 12x + y^2 - 2y + 12 = 0$ [A1]

(d) Explain why the tangent to the circle at Q is parallel to the y-axis. [2]

Let M be the centre of the circle, $M = (6,1)$

Equation of the radius, $MQ : y = 1$ is a horizontal line [B1]

Equation of tangent at $Q : x = 11$ is a vertical line [B1]

$\therefore$ The tangent to the circle at Q is parallel to the y-axis.

(e) Find the equation of the tangent to the circle at R. [2]

Gradient of $PR, m_{PR} = \dfrac{5-(-3)}{9-3}$

$= \dfrac{8}{6}$

$= \dfrac{4}{3}$

Gradient of the tangent at $R = -\dfrac{3}{4}$ [A1]

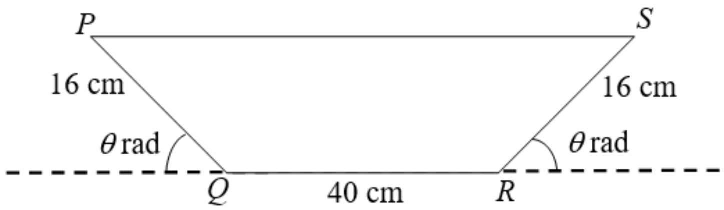
Equation of the tangent at R:

$\dfrac{y-5}{x-9} = -\dfrac{3}{4}$

$4y - 20 = -3x + 27$

$4y = -3x + 47$ [A1]

10 The diagram shows the vertical cross-section $PQRS$ of an open trough made from plastic sheeting. The lengths of PQ, QR and RS are 16 cm, 40 cm and 16 cm respectively. The trough rests with QR on horizontal ground and both PQ and RS are inclined at θ radians to the ground.



(a) Show that the area, A cm^2, of the cross-section $PQRS$ is given by [4]

$$A = 640\sin\theta + 128\sin 2\theta .$$

$PS = 40 + 2(16\cos\theta)$

$\quad = 40 + 32\cos\theta$ [A1]

Height of the trough $= 16\sin\theta$ [A1]

$A = \frac{1}{2}\left(40 + 40 + 32\cos\theta\right)(16\sin\theta)$

$\quad = 8\sin\theta(80 + 32\cos\theta)$

$\quad = 640\sin\theta + 256\sin\theta\cos\theta$ [A1]

$\quad = 640\sin\theta + 128(2\sin\theta\cos\theta)$

$\quad = 640\sin\theta + 128\sin 2\theta$ (shown) [A1]

(b) Given that θ can vary, find the value of θ for which the trough can hold a maximum amount of water. [5]

$A = 640\sin\theta + 128\sin 2\theta$

$\dfrac{dA}{d\theta} = 640\cos\theta + 256\cos 2\theta$

When $\dfrac{dA}{d\theta} = 0$,

$640\cos\theta + 256\cos 2\theta = 0$ [M1]

$5\cos\theta + 2\cos 2\theta = 0$

$5\cos\theta + 2(2\cos^2\theta - 1) = 0$

$4\cos^2\theta + 5\cos\theta - 2 = 0$

$\cos\theta = \dfrac{-5 \pm \sqrt{5^2 - 4(4)(-2)}}{2(4)}$ [M1]

$= \dfrac{-5 \pm \sqrt{57}}{8}$

$= \dfrac{-5 + \sqrt{57}}{8}$ or $\dfrac{-5 - \sqrt{57}}{8}$ (N.A. $\because -1 \le \cos\theta \le 1$)

$\theta = 1.246407756$

≈ 1.25 radians (to 3 s.f.) [A1]

$\dfrac{d^2A}{d\theta^2} = -640\sin\theta - 512\sin 2\theta$ [A1]

When $\theta = 1.246407756$,

$\dfrac{d^2A}{d\theta^2} = -915.9780788 \underline{\le 0}$,

$\therefore A$ is a maximum. [A1]