Name:	Class:	Class Register Number:

CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2023
SECONDARY 4

ADDITIONAL MATHEMATICS 4049/01

Paper 1 **Thursday 24 August 2023**

2 hours 15 minutes

Candidates answer on the Question Paper.

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
Total Marks	

This document consists of **19** printed pages and **1** blank page.

[Turn over

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 The points R and S have coordinates $\left(\sqrt{3}, 2\sqrt{3}\right)$ and $\left(\sqrt{5}, 4\sqrt{5}\right)$ respectively. Show that the gradient of RS can be expressed in the form $a+b\sqrt{15}$, where a and b are integers to be found. [4]

2 Given that $\cos\theta = p$ and that θ is acute, express in terms of p,

(a) $\sin\theta$, [1]

(b) $\tan\left(90^\circ - \theta\right)$, [2]

(c) $\cos 2\theta$. [2]

3 Express $\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}$ in partial fractions. [5]

4 The expression ax^3+bx^2+b leaves a remainder of R when divided by $(x+1)$ and a remainder of $5R-2$ when divided by $(x+2)$.

(a) Show that $b=\dfrac{2-3a}{5}$. [4]

(b) Given further that $ab=-8$ and $a>b$, find the value of a and of b. [3]

(c) Using the values of a and b found in **part (b)**, explain why the equation $ax^3 + bx^2 + b = 0$ has only one real root and state its value. [4]

5 Find the coordinates of the stationary points of the curve $y=\frac{(x-3)^2}{x}$ and determine the nature of each stationary point. [7]

6 The height of a ball above ground, h metres, released by a machine can be modelled by the equation $h = -0.2x^2 + 6x + 3$ where x is the horizontal distance travelled by the ball in metres.

(a) State the height above ground at which the ball is being released. [1]

(b) Express $h = -0.2x^2 + 6x + 3$ in the form $a(x-b)^2 + c$, where a, b and c are constants to be found. [2]

(c) Using your result from **part (b)**, explain why the height of the ball can never be more than 48 metres. [2]

(d) Hence, explain if this machine is safe for use in an indoor stadium with a ceiling height of 45 metres. [1]

7 **(a)** Solve the equation $3^x\left(18+3^x\right)=40$. [3]

(b) Solve the equation $\log_{\sqrt{2}} y = 3+\log_2\left(y+6\right)$. [5]

Continuation of working space for question 7(b)

(c) In order to obtain a graphical solution of the equation $x = 2\ln\left(4 - \frac{3x}{2}\right)$, a suitable straight line can be drawn on the same set of axes as the graph of $y = 4 - e^{\frac{x}{2}}$. Make $e^{\frac{x}{2}}$ the subject of $x = 2\ln\left(4 - \frac{3x}{2}\right)$ and hence find the equation of this line. [3]

8 A curve has the equation $y = \dfrac{x^2 - x + 1}{5x - 5}$, $x \neq 1$.

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 - 2x}{5(x-1)^2}$. [3]

(b) Explain why the curve is increasing for $x > 2$. [3]

9 **(a)** Given that $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{3}{4}$, prove that $\cos\alpha\cos\beta=7\sin\alpha\sin\beta$. [2]

(b) Hence, deduce the relationship between $\tan\alpha$ and $\tan\beta$. [2]

(c) Given further that $\alpha+\beta=45°$, calculate the value of $\tan\alpha+\tan\beta$. [3]

10 A circle, whose equation is $x^2 + y^2 - 10x - 8y + 16 = 0$, has centre C and radius r.

(a) Find the coordinates of C and the value of r. [3]

(b) Explain whether the point $(9, 2)$ lies inside or outside the circle. [2]

The line $4y = 3x + 1$ meets the circle at the points P and T, and the x-axis at S.
T lies between P and S.

(c) Without finding the coordinates of P and of T, find the length TS. [4]

11 The table shows, to 1 decimal place, the mass, m, of a radioactive substance, in grams, after t days .

t	5	10	15	20	25
m	57.7	37.0	23.7	15.2	9.7

(a) On the grid opposite, plot $\ln m$ against t and draw a straight line graph. [2]

(b) Find the gradient of your straight line and hence express m in the form $m_0 e^{kt}$, where m_0 and k are constants. [4]

The half-life of a radioactive substance is the length of time it takes for half of the substance to decay.

(c) In order to determine the half-life of the radioactive substance, a suitable straight line can be drawn on the same set of axes as your graph. Find the equation of this line and hence determine the half-life of the radioactive substance. [3]

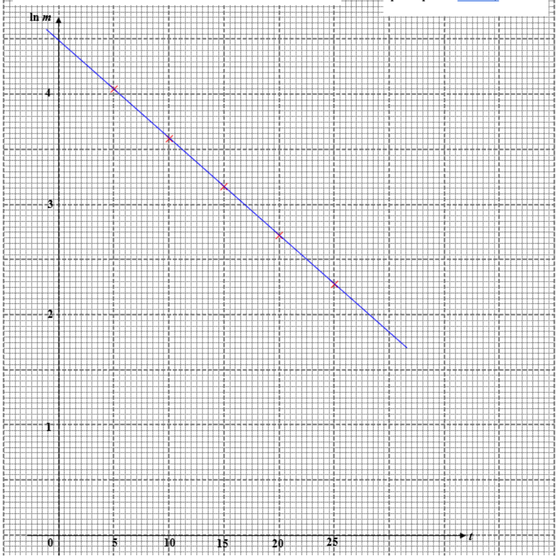
12 A particle travels in a straight line so that its velocity, v cm/s, t seconds after passing through a fixed point O, is given by $v = t^2 - kt + 5$, where k is a constant. The particle first comes to an instantaneous rest at the point P and then at the point Q.

(a) Given that the particle reaches a minimum velocity at $t = 3$, show that $k = 6$. [2]

(b) Find the distance PQ. [6]

(c) With working clearly shown, explain whether the particle will pass by O again, after the first 7 seconds. [2]

Answer Key

Qns	Ans	Qns	Ans
1	$7-3\sqrt{15}$	**10(a)**	$C(5,4)$, $r=5$
2(a)	$\sqrt{1-p^2}$	**10(b)**	Since distance to centre = $\sqrt{20}<5$ (radius) ,$(9,2)$ lies inside the circle.
2(b)	$\dfrac{p}{\sqrt{1-p^2}}$	**10(c)**	$1\frac{2}{3}$ units
2(c)	$2p^2-1$	**11(a)**	t: 5, 10, 15, 20, 25 $\ln m$: 4.055, 3.61, 3.16, 2.72, 2.27 P1 – at least 3 points P2 – straight line drawn with all points plotted correctly 
3	$\dfrac{8}{2x-1}+\dfrac{2}{x+2}-\dfrac{3}{(x+2)^2}$		
4(a)	Show qns		
4(b)	$a=4,\ b=-2$		
4(c)	For $2x^2+x+1=0$, discriminant $=1^2-4(2)(1)=-7<0$ $\therefore 2x^2+x+1=0$ has no real roots. Only 1 real root of $x=1$.		
5	$(3,0)$ is a minimum point $(-3,-12)$ is a maximum point		
6(a)	3 metres		
6(b)	$-0.2(x-15)^2+48$		
6(c)	$-0.2(x-15)^2+48\le 48$ Since max height = 48m, it will never exceed 48m.	**11(b)**	Gradient = -0.09 $m=90.0e^{-0.09t}$
6(d)	Not safe, maximum height of 48 m > 45 m.	**11(c)**	y = 3.8, half-life = 7.75 days
7(a)	x = 0.631 (3 s.f.)	**12(a)**	Show qns
7(b)	y = 6	**12(b)**	PQ = $10\frac{2}{3}$ m
7(c)	$y=\dfrac{3x}{2}$	**12(c)**	At $t=7,\ s=2\frac{1}{3},\ v=12$ Since $s>0,\ v>0$ and there are no more turning points after $t=5,$ the particle will not return to O after 7 seconds.
8(a)	Show qns		
8(b)	$x>2,\ x-2>0, x(x-2)>0$ since $(x-1)^2>0, \dfrac{x^2-2x}{5(x-1)^2}>0.$ Since $\dfrac{dy}{dx}>0,$ the curve is always increasing.		
9(a)	Show qns		
9(b)	$\tan\alpha\tan\beta=\dfrac{1}{7}$ or $\dfrac{1}{\tan\beta}=7\tan\alpha$		
9(c)	$\tan\alpha+\tan\beta=\dfrac{6}{7}$		

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中正中学

CHUNG CHENG HIGH SCHOOL (MAIN)

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2 hours 15 minutes

MARKS SCHEME

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where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1) \ldots (n-r+1)}{r!}$.

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1 The points R and S have coordinates $\left(\sqrt{3}, 2\sqrt{3}\right)$ and $\left(\sqrt{5}, 4\sqrt{5}\right)$ respectively. Show that the gradient of RS can be expressed in the form $a+b\sqrt{15}$, where a and b are integers to be found. [4]

Gradient of $RS = \dfrac{4\sqrt{5}-2\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$	B1 – $\dfrac{4\sqrt{5}-2\sqrt{3}}{\sqrt{5}-\sqrt{3}}$
$= \dfrac{\left(4\sqrt{5}-2\sqrt{3}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$	M1 – √ $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$
$= \dfrac{20-4\sqrt{15}-2\sqrt{15}-6}{2}$	M1 – either numerator or denominator expanded correctly
$= \dfrac{14-6\sqrt{15}}{2}$	
$= \dfrac{2\left(7-3\sqrt{15}\right)}{2}$	
$= 7-3\sqrt{15}$	
$a=7,\ b=-3$	A1

2 Given that $\cos\theta = p$ and that θ is acute, express in terms of p,

(a) $\sin\theta$, [1]

$\sqrt{1-p^2}$	B1

(b) $\tan\left(90^\circ-\theta\right)$, [2]

$=\dfrac{1}{\tan\theta}$	
$=\dfrac{1}{\dfrac{\sqrt{1-p^2}}{p}}$	M1 – $\dfrac{1}{\text{their } \tan\theta}$
$=\dfrac{p}{\sqrt{1-p^2}}$	A1

(c) $\cos 2\theta$. [2]

$\cos 2\theta = 1-2\sin^2\theta$	
$=1-2\left(\sqrt{1-p^2}\right)^2$	M1 – uses any $\cos 2\theta$ formula correctly
$=1-2\left(1-p^2\right)$	
$=1-2+2p^2$	
$=2p^2-1$	A1

3 Express $\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}$ in partial fractions. [5]

$\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}=\dfrac{A}{2x-1}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$	M1 – realising the form of partial fractions
$12x^2+32x+31=A(x+2)^2+B(2x-1)(x+2)+C(2x-1)$	M1 – realising the need to eliminate the denominator
let $x=\dfrac{1}{2}$, $50=\dfrac{25}{4}A$ $A=8$ let $x=-2$, $15=-5C$ $C=-3$ let $x=0$, $31=4A-2B-C$ $31=32-2B+3$ $2B=4$ $B=2$	
$A=8,\ B=2,\ C=-3$ $\dfrac{12x^2+32x+31}{(2x-1)(x+2)^2}=\dfrac{8}{2x-1}+\dfrac{2}{x+2}-\dfrac{3}{(x+2)^2}$	A1, A1, A1 – do not award last A1 if not expressed in partial fractions

4 The expression ax^3+bx^2+b leaves a remainder of R when divided by $(x+1)$ and a remainder of $5R-2$ when divided by $(x+2)$.

(a) Show that $b=\dfrac{2-3a}{5}$. [4]

when $x=-1$,	
$a(-1)^3+b(-1)^2+b=R$	M1 – realises $f(-1)=R$
$-a+2b=R$ --- (1)	
when $x=-2$,	
$a(-2)^3+b(-2)^2+b=5R-2$	M1 – realises $f(-2)=5R-2$
$-8a+5b=5R-2$ --- (2)	
sub (1) into (2),	
$-8a+5b=5(-a+2b)-2$	DM1 – award only if both M1 above is achieved
$-8a+5b=-5a+10b-2$	
$10b-5b=-8a+5a+2$	
$5b=-3a+2$	A1
$b=\dfrac{2-3a}{5}$ (shown)	AG

(b) Given further that $ab=-8$ and $a>b$, find the value of a and of b. [3]

$ab=-8$	
$b=\dfrac{-8}{a}$	
$\dfrac{-8}{a}=\dfrac{-3a+2}{5}$	M1 – allow slips only for LHS
$-40=-3a^2+2a$	
$3a^2-2a-40=0$	
$(3a+10)(a-4)=0$	
$a=-\dfrac{10}{3}$ or $a=4$	
$b=2.4$ $b=-2$	√M1 – finding ‘b’
(reject)	
$\therefore a=4,\ b=-2$	A1

(c) Using the values of a and b found in **part (b)**, explain why the equation $ax^3+bx^2+b=0$ has only one real root and state its value. [4]

$4x^3-2x^2-2=0$	
$2x^3-x^2-1=0$	
when $x=1$,	
$2(1)^3-1^2-1=0$	
$\therefore$ By factor theorem, $(x-1)$ is a factor.	B1 - factor
$(x-1)(2x^2+cx+1)=0$	M1 – correct method to find quadratic (division or inspection)
by comparing x^2 terms,	
$-2+c=-1$	
$c=1$	
$(x-1)(2x^2+x+1)=0$	
For $2x^2+x+1=0$,	
discriminant $=1^2-4(2)(1)=-7<0$	M1 – realising the need to find discriminant / solve quadratic equation
$\therefore 2x^2+x+1=0$ has no real roots.	
$4x^3-2x^2-2=0$ has only 1 real root of $x=1$.	A1 – no real roots + x = 1

5 Find the coordinates of the stationary points of the curve $y=\dfrac{(x-3)^2}{x}$ and determine the nature of each stationary point. [7]

$y=\dfrac{(x-3)^2}{x}$	
$=\dfrac{x^2-6x+9}{x}$	
$=x-6+\dfrac{9}{x}$	
$=x-6+9x^{-1}$	
$\dfrac{dy}{dx}=1-9x^{-2}$	B1
$=1-\dfrac{9}{x^2}$	
For stationary points, $\dfrac{dy}{dx}=0$	
$1-\dfrac{9}{x^2}=0$	M1 – sets $\dfrac{dy}{dx}$ to 0
$\dfrac{9}{x^2}=1$	
$x^2=9$	
$x=3$ or -3	A1 – for (3,0)
$y=0$ or -12	A1 – for (-3, -12)
$\dfrac{d^2y}{dx^2}=18x^{-3}$	M1 – their 2nd derivative or use of 1st derivative test
$=\dfrac{18}{x^3}$	
At $(3,0)$,	
$\dfrac{d^2y}{dx^2}=\dfrac{18}{3^3}=\dfrac{2}{3}>0$	
$\therefore (3,0)$ is a minimum point	A1
At $(-3,-12)$,	
$\dfrac{d^2y}{dx^2}=\dfrac{18}{(-3)^3}=-\dfrac{2}{3}<0$	
$\therefore (-3,-12)$ is a maximum point	A1

6 The height above ground, h metres, of a ball, released by a machine can be modelled by the equation $h = -0.2x^2 + 6x + 3$ where x is the horizontal distance travelled by the ball in metres.

(a) State the height above ground at which the ball is released. [1]

3 metres

(b) Express $h = -0.2x^2 + 6x + 3$ in the form $a(x-b)^2 + c$, where a, b and c are constants to be found. [2]

$h = -0.2(x^2 - 30x) + 3$	
$= -0.2(x^2 - 30x + 15^2 - 15^2) + 3$	
$= -0.2\left[(x-15)^2 - 225\right] + 3$	
$= -0.2(x-15)^2 + 48$	B2, 1 : –1 for each error

(c) Using your result from **(b)**, explain why the height of the ball can never be more than 48 metres. [2]

For $-0.2(x-15)^2 + 48$,	
$(x-15)^2 \geq 0$,	√ M1 – their square term
$-0.2(x-15)^2 \leq 0$	
$-0.2(x-15)^2 + 48 \leq 48$	
Since the ball reaches a maximum height of 48 m, it will never reach a height of more than 48 m.	A1

(d) Hence, explain if this machine is safe for use in an indoor stadium with a ceiling height of 45 metres. [1]

Since the ball can reach a maximum height of 48m which exceeds the ceiling height of 45 m, this machine is not safe for use in the indoor stadium.	B1 – to have comparison of maximum height with ceiling height.

7 **(a)** Solve the equation $3^x\left(18+3^x\right)=40$. [3]

$$3^x\left(18+3^x\right)=40$$
$$\left(3^x\right)^2+18\left(3^x\right)-40=0$$

Let $y = 3^x$.

$$y^2+18y-40=0$$
$$\left(y+20\right)\left(y-2\right)=0$$
$$y=-20 \quad \text{or} \quad y=2$$

M1 – solving of quadratic equation

$$3^x=2$$

M1 – taking ln

$$\ln 3^x=\ln 2$$
$$3^x=-20 \quad (\text{rej.} \because 3^x>0) \quad \text{or} \quad x\ln 3=\ln 2$$
$$x=\frac{\ln 2}{\ln 3}$$
$$=0.631 \ \ (3 \text{ sig. fig.})$$

A1

(b) Solve the equation $\log_{\sqrt{2}} y=3+\log_2\left(y+6\right)$. [5]

$$\log_{\sqrt{2}} y=3+\log_2\left(y+6\right)$$
$$\log_{\sqrt{2}} y-\log_2\left(y+6\right)=3$$
$$\frac{\log_2 y}{\log_2\sqrt{2}}-\log_2\left(y+6\right)=3$$

B1 – change of base law

$$\frac{\log_2 y}{\frac{1}{2}}-\log_2\left(y+6\right)=3$$

M1 – power law

$$2\log_2 y-\log_2\left(y+6\right)=3$$
$$\log_2\left(\frac{y^2}{y+6}\right)=3$$

M1 – quotient law

Comparing,

$$\frac{y^2}{y+6}=3$$

M1 – removal of log/comparing

$$y^2=3y+18$$
$$y^2-3y-18=0$$
$$\left(y-6\right)\left(y+3\right)=0$$
$$y=6 \quad \text{or} \quad -3 \ \ (\text{rej.} \therefore y>0)$$

A1

Continuation of working space for question 7(b)

(c) In order to obtain a graphical solution of the equation $x = 2\ln\left(4 - \frac{3x}{2}\right)$, a suitable straight line can be drawn on the same set of axes as the graph of $y = 4 - e^{\frac{x}{2}}$. Make $e^{\frac{x}{2}}$ the subject of $x = 2\ln\left(4 - \frac{3x}{2}\right)$ and hence find the equation of this line. [3]

$x = 2\ln\left(4 - \frac{3x}{2}\right)$	
$\frac{x}{2} = \ln\left(4 - \frac{3x}{2}\right)$	M1 – attempt to make $\ln\left(4 - \frac{3x}{2}\right)$ the subject
$e^{\frac{x}{2}} = 4 - \frac{3x}{2}$	A1
$\frac{3x}{2} = 4 - e^{\frac{x}{2}}$	
Equation of line: $y = \frac{3x}{2}$	B1

8 A curve has the equation $y=\frac{x^2-x+1}{5x-5},\ x\neq 1.$

(a) Show that $\frac{dy}{dx}=\frac{x^2-2x}{5(x-1)^2}$. [3]

$$\frac{dy}{dx}=\frac{(5x-5)(2x-1)-(x^2-x+1)(5)}{(5x-5)^2}$$
$$=\frac{10x^2-5x-10x+5-5x^2+5x-5}{[5(x-1)]^2}$$
$$=\frac{5x^2-10x}{5^2(x-1)^2}$$
$$=\frac{5(x^2-2x)}{25(x-1)^2}$$
$$=\frac{x^2-2x}{5(x-1)^2}\text{ (shown)}$$

M1 – quotient rule performed correctly, allow for numerical slips

B1 – for $5^2(x-1)^2/25(x-1)^2$

A1 – factorisation must be seen

AG

(b) Explain why the curve is increasing for $x>2$. [3]

<u>Alternative solution</u>

Where the curve is increasing,

$\frac{dy}{dx}>0,$

$\frac{x^2-2x}{5(x-1)^2}>0$

Since $5(x-1)^2>0$, for all real values of x,

$x^2-2x>0$

$x(x-2)>0$

$x<0$ or $x>2$

$\therefore$ for $x>2,$ the curve is always increasing.

$x^2-2x=x(x-2)$

$x>2,\ x-2>0$

$x(x-2)>0$

since $(x-1)^2>0$

$\frac{x^2-2x}{5(x-1)^2}>0$

Since $\frac{dy}{dx}>0,$ the curve is always increasing.

B1 – $x>2,\ x-2>0,\ x(x-2)>0$

B1 – $(x-1)^2>0$

B1 – $\frac{x^2-2x}{5(x-1)^2}>0$ + conclusion of curve is always increasing. Award only if B1,B1 has been achieved.

9 **(a)** Given that $\dfrac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\dfrac{3}{4}$, prove that $\cos\alpha\cos\beta=7\sin\alpha\sin\beta$. [2]

$4[\cos\alpha\cos\beta-\sin\alpha\sin\beta]=3[\cos\alpha\cos\beta+\sin\alpha\sin\beta]$	M1 – attempt at addition formula
$4\cos\alpha\cos\beta-4\sin\alpha\sin\beta=3\cos\alpha\cos\beta+3\sin\alpha\sin\beta$	
$4\cos\alpha\cos\beta-3\cos\alpha\cos\beta=4\sin\alpha\sin\beta+3\sin\alpha\sin\beta$	A1
$\cos\alpha\cos\beta=7\sin\alpha\sin\beta$ (shown)	AG

(b) Hence, deduce the relationship between $\tan\alpha$ and $\tan\beta$. [2]

$\cos\alpha\cos\beta=7\sin\alpha\sin\beta$	
$\dfrac{1}{7}=\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$	M1 – realises the need to divide
$\tan\alpha\tan\beta=\dfrac{1}{7}$ or $\dfrac{1}{\tan\beta}=7\tan\alpha$	A1

(c) Given further that $\alpha+\beta=45°$, calculate the value of $\tan\alpha+\tan\beta$. [3]

$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$	M1 – $\tan 45°$ or realises the need for $\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$\tan 45°=\dfrac{\tan\alpha+\tan\beta}{1-\frac{1}{7}}$	M1- their $\tan\alpha\tan\beta$
$1=\dfrac{\tan\alpha+\tan\beta}{\frac{6}{7}}$	
$\tan\alpha+\tan\beta=\dfrac{6}{7}$	A1

10 A circle, whose equation is $x^2 + y^2 - 10x - 8y + 16 = 0$, has centre C and radius r.

(a) Find the coordinates of C and the value of r. [3]

$-2a = -10$	
$a = 5$	
$-2b = -8$	
$b = 4$	
$a^2 + b^2 - r^2 = 16$	
$r^2 = 5^2 + 4^2 - 16$	M1 – their a and b
$r = 5$	
$\therefore C(5,4)$ and r is 5 units.	B1 – (5,4), A1 – 5 units
Alternative Solution	
$x^2 + y^2 - 10x - 8y + 16 = 0$	
$x^2 - 10x + 5^2 - 5^2 + y^2 - 8y + 4^2 - 4^2 + 16 = 0$	M1 – attempt to complete square
$(x-5)^2 + (y-4)^2 = 25 + 16 - 16$	
$(x-5)^2 + (y-4)^2 = 5^2$	
$C(5,4)$ and $r = 5$ units.	B1 – (5,4), A1 – 5 units

(b) Explain whether the point $(9,2)$ lies inside or outside the circle. [2]

Distance from $(9,2)$ to C	
$= \sqrt{(9-5)^2 + (4-2)^2}$	M1 – realises the need to find distance from (9,2) to C
$= \sqrt{20} < 5$ (radius)	A1 – need to indicate < 5 (radius) and therefore inside the circle
$\therefore (9,2)$ lies inside the circle.	

The line $4y = 3x + 1$ meets the circle at the points *P* and *T*, and the *x*-axis at *S*.
T lies between *P* and *S*.

(c) Without finding the coordinates of *P* and of *T*, find the length *TS*. [4]

At $S,\ y = 0$	M1 – realises that $y = 0$ for *S*
$3x + 1 = 0$	
$x = -\dfrac{1}{3}$	
$S\left(-\dfrac{1}{3}, 0\right)$	
Subst $x = 5$ into $4y = 3x + 1$,	B1 – shows centre lies on line
$4y = 3(5) + 1$	
$y = \dfrac{16}{4} = 4$	
$\therefore (5, 4)$ lies on the line $4y = 3x + 1$	
Distance from C to S	
$= \sqrt{\left(5 - \left(-\dfrac{1}{3}\right)\right)^2 + 4^2}$	M1 – distance from their *C* to *S*
$= 6\dfrac{2}{3}$ units	
$\therefore TS = 6\dfrac{2}{3} - 5$	
$= 1\dfrac{2}{3}$ units	A1

11 The table shows, to 1 decimal place, the mass, m of a radioactive substance, in grams, after t days.

t	5	10	15	20	25
m	57.7	37.0	23.7	15.2	9.7

(a) On the grid opposite, plot $\ln m$ against t and draw a straight line graph. [2]

(b) Find the gradient of your straight line and hence express m in the form $m_0 e^{kt}$, where m_0 and k are constants. [4]

gradient = $-\frac{2.25}{25} = -0.09$	B1 – finds gradient correctly
$\ln m = -0.09t + 4.5$	M1 – finds $\ln m = 'm't + c$ with their gradient + y -intercept
$m = e^{-0.09t+4.5}$	
$m = e^{-0.09t} \bullet e^{4.5}$	
$m = 90.0e^{-0.09t}$	A1,A1

The half-life of a radioactive substance is the length of time it takes for half of the substance to decay.

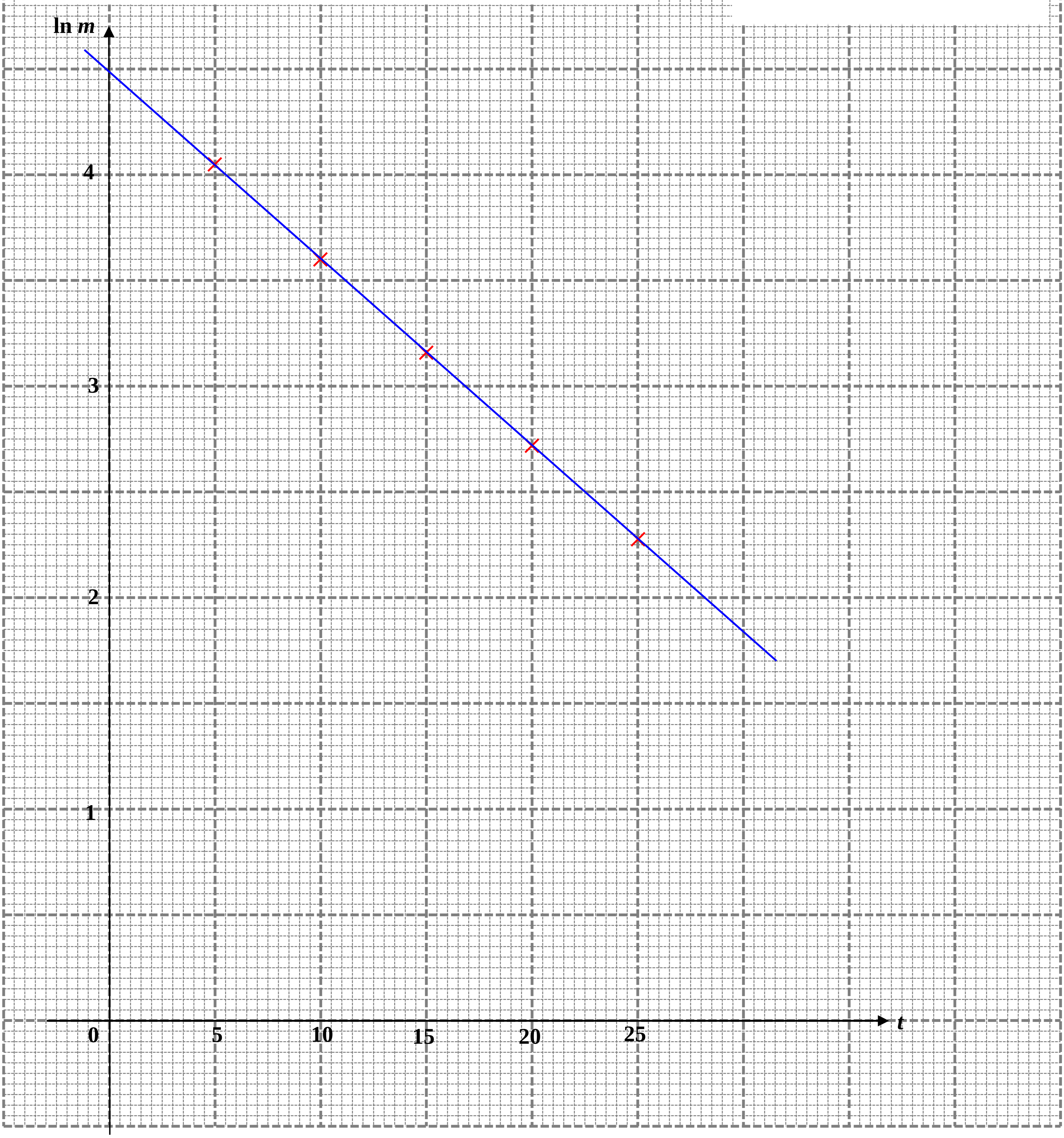
(c) In order to determine the half-life of the radioactive substance, a suitable straight line can be drawn on the same set of axes as your graph. Find the equation of this line and hence determine the half-life of the radioactive substance. [3]

$m = 90.0e^{-0.09t}$	
when $t = 0$,	
$m = 90.0$ (inital mass)	M1 – attempt to find original mass
half of original mass $= 45$	
$\ln 45 = 3.80.$	
line to be drawn :	
$y = 3.8$	A1
Half-life = 7.75 days	A1

t	5	10	15	20	25
$\ln m$	4.055	3.61	3.16	2.72	2.27

P1 – at least 3 points

P2 – straight line drawn with all points plotted correctly



12 A particle travels in a straight line so that its velocity, v cm/s, t seconds after passing through a fixed point O, is given by $v = t^2 - kt + 5$, where k is a constant. The particle first comes to an instantaneous rest at the point P and then at the point Q.

(a) Given that the particle reaches a minimum velocity at $t = 3$, show that $k = 6$. [2]

$v = t^2 - kt + 6$

$= \left(t - \frac{k}{2}\right)^2 + 5 - \frac{k^2}{4}$ $\boxed{\text{B1}}$

minimum velocity occurs

when $t - \frac{k}{2} = 0$

$3 - \frac{k}{2} = 0$ $\boxed{\sqrt{}\text{M1}}$

$6 - k = 0$

$k = 6$ (shown)

$v = t^2 - kt + 6$	
$\frac{dv}{dt} = 2t - k$	B1
when $t = 3$, $\frac{dv}{dt} = 0$	√M1 – recognises $\frac{dv}{dt} = 0$ when $t = 3$
$2(3) - k = 0$	
$6 - k = 0$	
$k = 6$ (shown)	AG

(b) Find the distance PQ. [6]

At P and Q, $v = 0$	
$t^2 - 6t + 5 = 0$	M1 – equates v to 0
$(t-1)(t-5) = 0$	
$t = 1$ or $t = 5$	A1
$s = \frac{1}{3}t^3 - 3t^2 + 5t + c$	M1 – realises the need to integrate
when $t = 0, s = 0$, $\therefore c = 0$	
$s = \frac{1}{3}t^3 - 3t^2 + 5t$	A1
when $t = 1, s = 2\frac{1}{3}$	M1 – s at their $t = 1$ or $t = 5$
when $t = 5, s = -8\frac{1}{3}$	
Distance $PQ = 2\frac{1}{3} + 8\frac{1}{3}$	
$= 10\frac{2}{3}$ m	A1

(c) With working clearly shown, explain whether the particle will pass by O again, after the first 7 seconds.

[2]

At $t = 7,\ s = 2\frac{1}{3},\ v = 12$	M1 – finds s or v at $t = 7$
Since $s > 0,\ v > 0$ and there are no more turning points after $t = 5,$ the particle will not return to O after 7 seconds.	A1 - explains no turning points particle will not return to O

Alternative solution

$\frac{1}{3}t^3 - 3t^2 + 5t = 0$ $t^3 - 9t^2 + 15t = 0$ $t\left(t^2 - 9t + 15\right) = 0$ $t = 0$ or $t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(15)}}{2}$ $= 6.79$ or 2.21	M1 – Attempt at solution for $s = 0$
Since the last time that the particle is at O is 6.79s, which is before 7 seconds, the particle will not pass by O again after 7 seconds.	A1 – explains that last time that the particle is at O is 6.79s , which is before 7s and hence does not return to O

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Parent's Signature

PRELIMINARY EXAMINATION 2023
SECONDARY 4

ADDITIONAL MATHEMATICS — 4049/02

Paper 2 — **Tuesday 29 August 2023**

Candidates answer on the Question Paper. — **2 hours 15 minutes**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use	
Question Number	Marks Obtained
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total Marks	

This document consists of **20** printed pages and **2** blank pages.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2+bx+c=0$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Binomial expansion

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{r}a^{n-r}b^r+\ldots+b^n,$$

where n is a positive integer and $\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A+\cos^2 A=1$$

$$\sec^2 A=1+\tan^2 A$$

$$\operatorname{cosec}^2 A=1+\cot^2 A$$

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$

$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

$$\sin 2A=2\sin A\cos A$$

$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$a^2=b^2+c^2-2bc\cos A$$

$$\Delta=\frac{1}{2}ab\sin C$$

1 **(a)** The line $4x = 3y + 2$ intersects the curve $x^2 - xy + 5 = 0$ at the points A and B.
Find the midpoint of AB. [5]

(b) Find the least value of the integer h for which hx^2+5x+h is positive for all real values of x. [3]

(c) Given that the line $y=3x+p$ is tangent to the curve $y=x^2+5x+q$, where p and q are integers, prove that p and q are consecutive numbers. [4]

2 **(a)** By considering the general term in the binomial expansion of $\left(x^3-\frac{2}{x}\right)^7$, explain why there are only odd powers of x in this expansion. [3]

(b) Find the term independent of x in the expansion of $\left(x^3-\frac{2}{x}\right)^7\left(\frac{5}{x}-2x^2\right)$. [3]

3 The expression $7\sin\theta + 3\cos\theta$ is defined for $0° \le \theta \le 360°$.

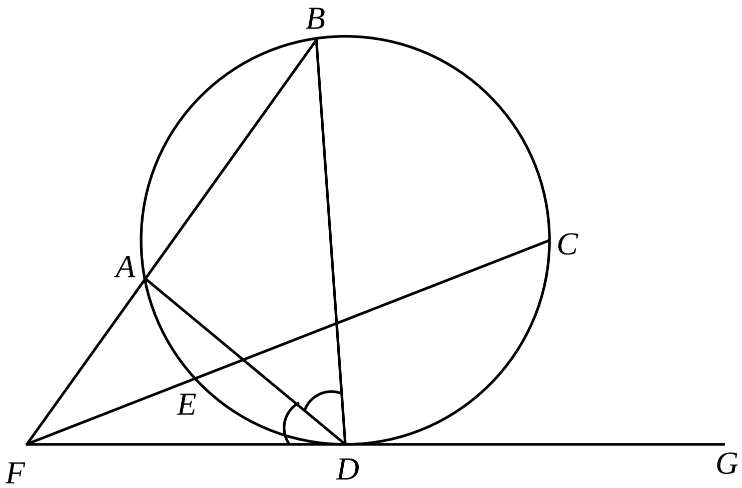
(a) Using $R\sin(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, solve the equation

$$7\sin\theta = 5 - 3\cos\theta\,.$$

[5]

(b) State the largest and smallest values of $\left(7\sin\theta+3\cos\theta\right)^2-12$ and find the corresponding values of θ. [4]

4



The diagram shows a circle passing through the points A, B, C, D and E. The straight line FDG is tangent to the circle at D while FAB and FEC are secant lines.
Given that angle FDA = angle ADB,

(a) show that triangle ABD is an isosceles triangle, [2]

(b) prove that $AF \times BF = EF \times CF$. [3]

5 An object is heated in an oven until it reaches a temperature of X °C. It is then allowed to cool under room temperature. Its temperature, T °C, can be modelled by $T = 18 + 62e^{-kt}$, where t is the time in minutes since the object starts cooling.

(a) Find the value of X. [1]

When t = 10, the temperature of the object is 65 °C.

(b) Find the temperature of the object an hour later, giving your answer to one decimal place. [5]

(c) What does the model suggest about the room temperature? Explain your answer. [2]

6 The equation of a curve is $y = \ln\left(\dfrac{2x-1}{3x-1}\right)$, where $x > \dfrac{1}{2}$.

(a) Find $\dfrac{dy}{dx}$, expressing it as a single fraction. [3]

(b) Explain why the curve will be almost parallel to the x-axis as x becomes very large. [2]

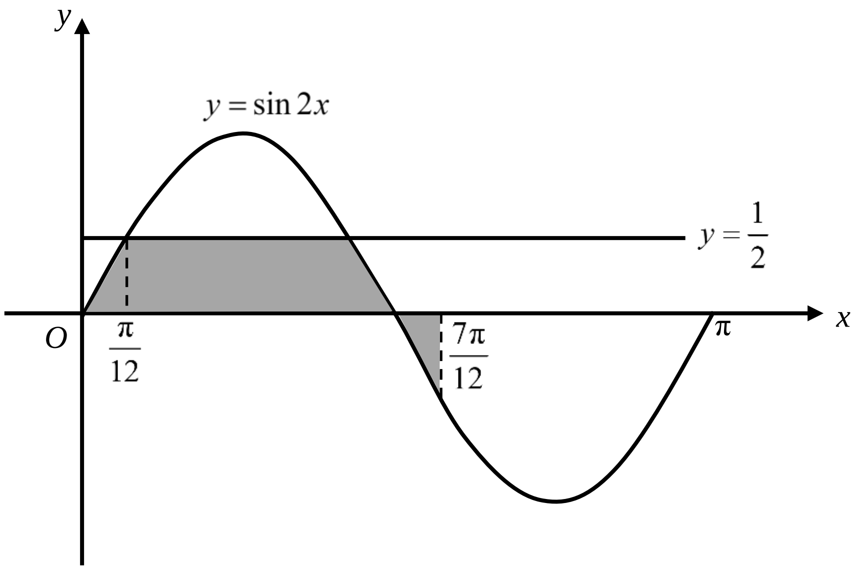
(c) Find the value of x at the instant when the rate of change of x is twice the rate of change of y. [3]

7 **(a)** Show that $\tan A + \cot A = 2\operatorname{cosec} 2A$. [3]

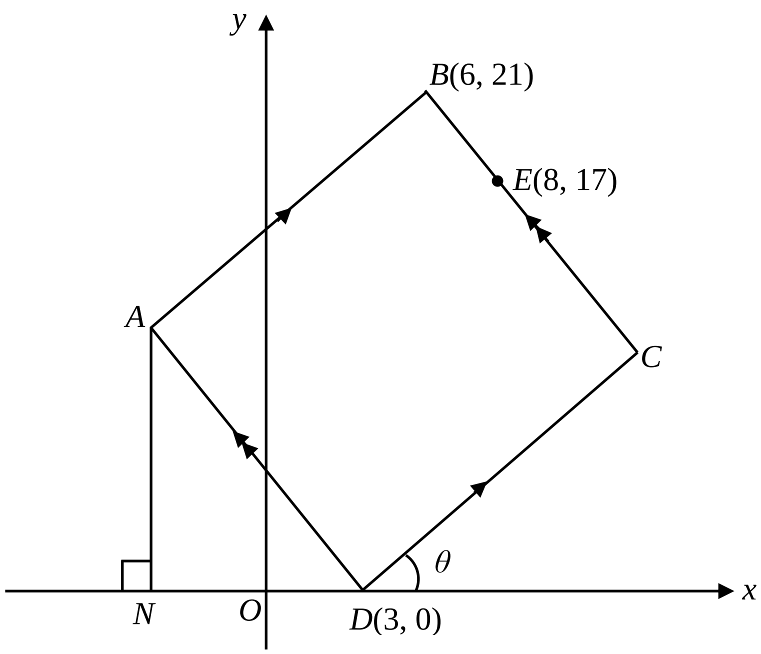
(b) Hence, solve the equation $\dfrac{1}{\tan A + \cot A} = \dfrac{1}{4}$ for $0 \le A \le 2\pi$. [3]

(c) The diagram shows, for $0 \le x \le \pi$, the curve $y = \sin 2x$ and the line $y = \frac{1}{2}$.

Showing all your working, find the area of the shaded region. [5]



8



The diagram shows a parallelogram with vertices A, $B(6, 21)$, C and $D(3, 0)$. The point $E(8, 17)$ lies on BC. The line CD makes an angle θ with the positive x-axis such that $\tan\theta = 1$. A line is drawn, parallel to the y-axis, from A to meet the x-axis at N.

(a) Show that the coordinates of A are $(-3, 12)$. [5]

(b) Hence, find the area of parallelogram *ABCD*. [2]

(c) A point *F* with y-coordinate of 5 lies on the line *CD*.
Explain why *AEFN* is a parallelogram. [5]

9 **(a)** Differentiate $x \ln x$ with respect to x. [2]

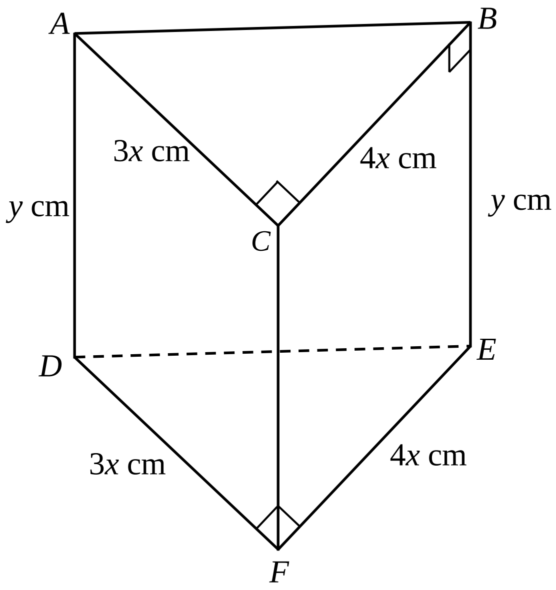
(b) A curve $y = \mathrm{f}(x)$ is such that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 24x^2 + \frac{16}{x}$, where $x > 0$. The line $y = 24x - 40$ is parallel to the tangent of the curve at $P(1, -16)$.

By using the result found in part **(a)**, find the equation of the curve. [6]

Continuation of working space for question **9(b)**.

(c) Explain why the condition $x > 0$ is necessary. [1]

10



The diagram shows a solid prism with right-angled triangular ends that are perpendicular to the parallel sides *AD*, *BE* and *CF*, which are each y cm in length. The right-angled triangular ends have sides *AC* and *DF*, which are $3x$ cm, and sides *BC* and *EF*, which are $4x$ cm.

Given that the volume of the prism is 1200 cm^3,

(a) find an expression for y in terms of x, [2]

(b) show that the total surface area of the prism, S cm^2, is given by $S = 12x^2 + \frac{2400}{x}$. [3]

(c) Given that x can vary, find the value of x for which the total surface area of the prism is a stationary value. [3]

(d) Explain why this value of x gives the smallest surface area possible. [2]

Answer Key

Qns	Ans	Qns	Ans
1(a)	$\left(1,\ \frac{2}{3}\right)$	**8(a)**	Show qns
1(b)	$h = 3$	**8(b)**	162 units^2
1(c)	$q = 1 + p$, therefore consecutive numbers	**8(c)**	Since gradients of lines AE and FN are the same, the lines AE and FN are parallel. Since F lies directly below E, the line EF is a vertical line. Thus, the lines EF and AN are parallel. Since there are 2 pairs of parallel lines (or equivalent), $AEFN$ is a parallelogram.
2(a)	Power of $x = 21 - 4r$. Since **$4r$ is an even number** for all non-negative integer values of r and 21 is **an odd number**, then $21 - 4r$ is always an **odd number.**	**9(a)**	$1 + \ln x$
2(b)	-3360	**9(b)**	$y = 2x^4 + 16x\ln x - 18$
3(a)	$\theta = 17.8°$ or $115.8°$ (1 d.p.)	**9(c)**	$\ln x$ is defined for $x > 0$ or $\ln x$ is undefined for $x < 0$.
3(b)	Largest value $= 46$ when $\theta = 66.8°$ or $246.8°$ (1 d.p.) Smallest value $= -12$ when $\theta = 156.8°$ or $336.8°$ (1 d.p.)	**10(a)**	$y = \frac{200}{x^2}$
4(a)	Show qns	**10(b)**	
4(b)	Show qns	**10(c)**	$x = 4.64$
5(a)	80	**10(d)**	$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = 72 > 0$ $\because$ minimum
5(b)	For $t = 60$, $T = 29.8$°C For $t = 70$, $T = 26.9$°C		
5(c)	Room temp. = 18°, $62e^{-kt}$ approaches zero as t becomes larger.		
6(a)	$\frac{1}{(2x-1)(3x+1)}$		
6(b)	Since $(2x-1)(3x+1)$ becomes a very large number as x becomes very large, $\frac{\mathrm{d}y}{\mathrm{d}x}$ approaches 0, the curve will almost horizontal, thus it will be almost parallel to the x-axis.		
6(c)	$x = 1$		
7(a)	Show qns		
7(b)	$A = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}$ or $\frac{17\pi}{12}$		
7(c)	0.725 units^2		

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CHUNG CHENG HIGH SCHOOL (MAIN)

PRELIMINARY EXAMINATION 2023
SECONDARY 4

ADDITIONAL MATHEMATICS 4049/02

Paper 2 **Tuesday 29 August 2023**

Candidates answer on the Question Paper. **2 hours 15 minutes**

MARKS SCHEME

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For the equation $ax^2 + bx + c = 0$,

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$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} ab \sin C$$

1 **(a)** The line $4x = 3y + 2$ intersects the curve $x^2 - xy + 5 = 0$ at the points A and B. Find the midpoint of AB. [5]

$4x = 3y + 2 \quad ...(1)$	
$x^2 - xy + 5 = 0 \quad ...(2)$	
From (1):	
$3y = 4x - 2$	
$y = \dfrac{4x-2}{3} \quad ...(3)$	
Sub. (3) into (2):	
$x^2 - x\left(\dfrac{4x-2}{3}\right) + 5 = 0$	M1 f.t. – substitution
$3x^2 - 4x^2 + 2x + 15 = 0$	
$-x^2 + 2x + 15 = 0$	
$x^2 - 2x - 15 = 0$	
$(x-5)(x+3) = 0$	M1 f.t. – solving quadratic
$x = 5$ or -3 sub. into (3):	
$y = 6$ or $-\dfrac{14}{3}$	M1 – either correct x or y values
$\therefore A\left(-3, -\dfrac{14}{3}\right)$ **and** B**(5, 6)**	
Midpoint of $AB = \left(\dfrac{-3+5}{2}, \dfrac{-\frac{14}{3}+6}{2}\right)$	M1 f.t. – midpoint formula
$= \left(1, \dfrac{2}{3}\right)$	A1

(b) Find the least value of the integer h for which hx^2+5x+h is positive for all real values of x. [3]

For $hx^2+5x+h>0$,	
discriminant <0	
$25-4(h)h<0$	B1 – discriminant
$25-4h^2<0$	
$(5-2h)(5+2h)<0$	M1 f.t. – factorising quadratic
$h<-\frac{5}{2}$ or $h>\frac{5}{2}$	
Since $h>0$, $h>\frac{5}{2}$.	
$\therefore$ **Least integer value of h = 3.**	A1

(c) Given that the line $y=3x+p$ is tangent to the curve $y=x^2+5x+q$, where p and q are integers, prove that p and q are consecutive numbers. [4]

$y=3x+p$ ... (1)	
$y=x^2+5x+q$ ... (2)	
Sub. (1) into (2):	
$3x+p=x^2+5x+q$	M1 – substitution
$x^2+5x+q-3x-p=0$	
$x^2+2x+q-p=0$	M1 – forming quadratic
a = 1, b = 2, c = k – c	
Line is tangent to curve $\rightarrow$ $b^2-4ac=0$	
$2^2-4(1)(q-p)=0$	M1 f.t. – any use of discriminant
$4-4q+4p=0$	
$4q=4+4p$	
$q=1+p$	A1 – with explanation
Since q = 1 + p, q will always be the next number after p. Hence, p and q are consecutive numbers (proved).	

<u>Alternative:</u>

$y = x^2 + 5x + q$	
$\frac{dy}{dx} = 2x + 5$	
Since line is tangent to curve and gradient of line = 3,	
$2x + 5 = 3$	M1 – equate $\frac{dy}{dx}$ to 3
$2x = -2$	
$x = -1$	M1 f.t. – finding x
$y = 3x + p$ ... (1)	
$y = x^2 + 5x + q$ ... (2)	
Sub. (1) into (2) and $x = -1$:	
$3x + p = x^2 + 5x + q$	M1 – substitution
$3(-1) + p = (-1)^2 + 5(-1) + q$	
$-3 + p = -4 + q$	
$q - p = 1$	
Since the difference between q and p is 1, q will always be the next number after p. Hence, p and q are consecutive numbers (proved).	A1 – with explanation

2 **(a)** By considering the general term in the binomial expansion of $\left(x^3 - \frac{2}{x}\right)^7$, explain why there are only odd powers of x in this expansion. [3]

$T_{r+1} = \binom{7}{r}\left(x^3\right)^{7-r}\left(-\frac{2}{x}\right)^r$	B1 – general formula
$= \binom{7}{r}\left(x^{21-3r}\right)(-2)^r(x)^{-r}$	
$= \binom{7}{r}(-2)^r x^{21-4r}$	
Power of x = 21 – 4r	M1 f.t. – finding powers of x
Since **4*r* is an even number** for all non-negative integer values of r and 21 is an odd number, then **21 – 4*r* is always an odd number**.	A1 – conclusion
Therefore, there are only odd powers of x in this expansion.	

(b) Find the term independent of x in the expansion of $\left(x^3 - \frac{2}{x}\right)^7\left(\frac{5}{x} - 2x^2\right)$. [3]

Consider $21-4r=1$, $4r=20$, $r=5$	M1 f.t. – identifying x term
$\left(x^3-\frac{2}{x}\right)^7\left(\frac{5}{x}-2x^2\right)=\left[\ldots+\binom{7}{5}(-2)^5x^{21-4(5)}+\ldots\right]\left(\frac{5}{x}-2x^2\right)$ $=(\ldots-672x+\ldots)\left(\frac{5}{x}-2x^2\right)$	M1 f.t. – expansion
Term independent of $x=-672\times5$ $=-3360$	A1

3 The expression $7\sin\theta + 3\cos\theta$ is defined for $0° \le \theta \le 360°$.

(a) Using $R\sin(\theta+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$, solve the equation

$$7\sin\theta = 5 - 3\cos\theta\,.$$

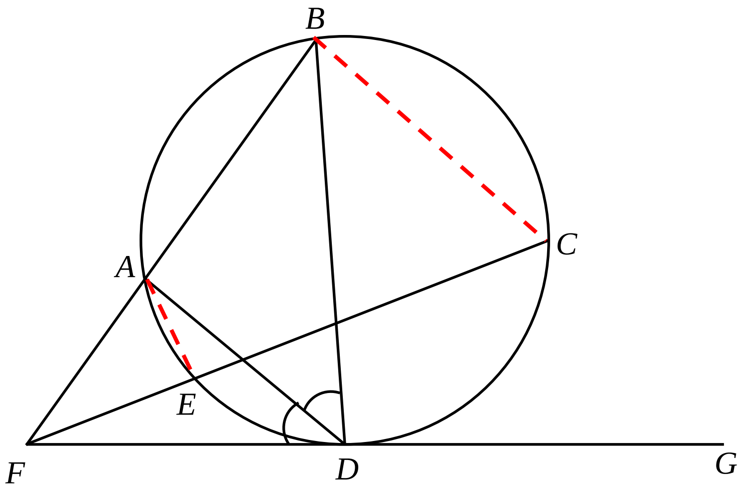
[5]

$7\sin\theta + 3\cos\theta = R\sin(\theta+\alpha)$	
$= R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$	
Comparing,	
$7 = R\cos\alpha$ and $3 = R\sin\alpha$	
$R = \sqrt{7^2+3^2}$	M1 – attempt to find R
$= \sqrt{58}$	
$\tan\alpha = \frac{3}{7}$	
$\alpha = \tan^{-1}\left(\frac{3}{7}\right)$	M1 – attempt to find α
$= 23.1985...°$	[only accept $\tan^{-1}\left(\frac{3}{7}\right)$ or $\tan^{-1}\left(\frac{7}{3}\right)$]
$\therefore 7\sin\theta + 3\cos\theta = \sqrt{58}\sin(\theta + 23.1985...°)$	
$7\sin\theta = 5 - 3\cos\theta$	
$7\sin\theta + 3\cos\theta = 5$	
$\sqrt{58}\sin(\theta + 23.1985...°) = 5$	M1 f.t. – substitute R-form
$\sin(\theta + 23.1985...°) = \frac{5}{\sqrt{58}}$	
basic angle $= \sin^{-1}\left(\frac{5}{\sqrt{58}}\right)$	M1 f.t. – basic angle
$= 41.0359...°$	
$\theta + 23.1985...° = 41.0359...°$ or $180° - 41.0359...°$	
$\theta = 17.8°$ or $115.8°$ (1 dec. pl.)	A1

(b) State the largest and smallest values of $\left(7\sin\theta+3\cos\theta\right)^2-12$ and find the corresponding values of θ. [4]

largest value of $\left(7\sin\theta+3\cos\theta\right)^2-12=\left(\sqrt{58}\right)^2-12$ or $\left(-\sqrt{58}\right)^2-12$ $=46$	B1
occurs when $\theta+23.1985...^\circ=90^\circ$ or 270° $\theta=66.8^\circ$ or 246.8° (1 dec. pl.)	B1
smallest value of $\left(7\sin\theta+3\cos\theta\right)^2-12=\left(0\right)^2-12$ $=-12$	B1
occurs when $\theta+23.1985...^\circ=180^\circ$ or 360° $\theta=156.8^\circ$ or 336.8° (1 dec. pl.)	B1

4



The diagram shows a circle passing through the points *A*, *B*, *C*, *D* and *E*. The straight line *FDG* is tangent to the circle at *D* while *FAB* and *FEC* are secant lines.
Given that angle *FDA* = angle *ADB*,

(a) show that triangle *ABD* is an isosceles triangle, [2]

$\angle ABD = \angle FDA$ (**alternate segment theorem**) B1

$= \angle ADB$

Since $\angle ABD = \angle ADB$, they form **base angles of isosceles triangle**. B1

Thus, triangle *ABD* is an isosceles triangle. (shown)

(b) prove that $AF \times BF = EF \times CF$. [3]

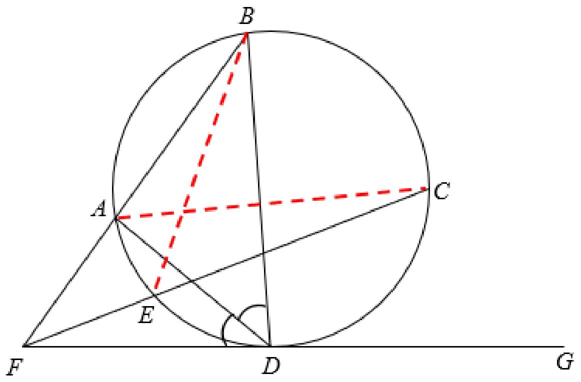
$\angle FAE = \angle FCB$ (**exterior $\angle$ of cyclic quadrilateral**) B1

$\angle AFE = \angle CFB$ (**common $\angle$**) B1

Thus, triangle *AFE* is similar to triangle *CFB*.

$\frac{AF}{CF} = \frac{EF}{BF}$ (**ratio of corresponding sides are equal**) B1

$AF \times BF = EF \times CF$ (proved)



Alternative:

$\angle FBE = \angle FCA$ (**$\angle$s in same segment**) B1

$\angle EFB = \angle AFC$ (**common $\angle$**) B1

Thus, triangle *EFB* is similar to triangle *AFC*.

$$\frac{AF}{EF} = \frac{CF}{BF}$$ (**ratio of corresponding sides are equal**) B1

$AF \times BF = EF \times CF$ (proved)

5 An object is heated in an oven until it reaches a temperature of X °C. It is then allowed to cool under room temperature. Its temperature, T °C, can be modelled by $T = 18 + 62e^{-kt}$, where t is the time in minutes since the object starts cooling.

(a) Find the value of X. [1]

When $t = 0$, $T = X$.

$$X = 18 + 62e^{-k(0)}$$
$$= 18 + 62$$
$$= 80$$

B1

When $t = 10$, the temperature of the object is 65 °C.

(b) Find the temperature of the object an hour later, giving your answer to one decimal place. [5]

When $t = 10$, $T = 65$.

$$65 = 18 + 62e^{-k(10)}$$

B1 – sub. 10 and 65 correctly

$$\frac{47}{62} = e^{-10k}$$
$$\ln\frac{47}{62} = \ln e^{-10k}$$

M1 – Take ln on both sides

$$\ln\frac{47}{62} = -10k$$
$$k = -\frac{1}{10}\ln\frac{47}{62}$$
$$= 0.0276986...$$

M1 – finding k

When $t = 60$,

$$T = 18 + 62e^{-\left(-\frac{1}{10}\ln\frac{47}{62}\right)(60)}$$
$$= 29.765...$$
$$= 29.8 \text{ °C (1 dec. pl.)}$$

When $t = 70$,

$$T = 18 + 62e^{-\left(-\frac{1}{10}\ln\frac{47}{62}\right)(70)}$$
$$= 26.9193...$$
$$= 26.9 \text{ °C (1 dec. pl.)}$$

M1 f.t.– sub. $t = 60$ or 70

A1

(c) What does the model suggest about the room temperature? Explain your answer. [2]

Since $e^{-kt} > 0$, then $62e^{-kt} > 0$.
Thus, **as t becomes larger**, **$62e^{-kt}$ approaches zero**.
Therefore, T will **approach 18 °C** as t becomes larger.
Hence, the room temperature is **18 °C.**

B1 – explanation

B1 – temperature

6 The equation of a curve is $y=\ln\left(\dfrac{2x-1}{3x-1}\right)$, where $x>\dfrac{1}{2}$.

(a) Find $\dfrac{dy}{dx}$, expressing it as a single fraction. [3]

$y=\ln\left(\dfrac{2x-1}{3x-1}\right)$	
$=\ln(2x-1)-\ln(3x-1)$	B1 – quotient law of logarithms
$\dfrac{dy}{dx}=\dfrac{2}{2x-1}-\dfrac{3}{3x+1}$	M1 f.t. – differentiate either term
$=\dfrac{2(3x-1)-3(2x-1)}{(2x-1)(3x-1)}$	
$=\dfrac{1}{(2x-1)(3x+1)}$	A1 – single fraction
<u>Alternative:</u>	
$\dfrac{dy}{dx}=\dfrac{\dfrac{2(3x-1)-3(2x-1)}{(3x-1)^2}}{\dfrac{2x-1}{3x-1}}$	B1 – differentiate ln in the form $\dfrac{f'(x)}{f(x)}$ M1 – apply quotient rule to numerator.
$=\dfrac{\dfrac{1}{(3x-1)^2}}{\dfrac{2x-1}{3x-1}}$	
$=\dfrac{1}{(2x-1)(3x+1)}$	A1 – single fraction

(b) Explain why the curve will be almost parallel to the x-axis as x becomes very large. [2]

Since $(2x-1)(3x+1)$ becomes a very large number as x becomes very large, $\dfrac{dy}{dx}$ approaches **<u>0</u>**,	M1
the curve will **<u>almost horizontal</u>**, thus it will be almost parallel to the x-axis.	A1

(c) Find the value of x at the instant when the rate of change of x is twice the rate of change of y. [3]

$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$	
$\frac{dy}{dt}=\frac{1}{(2x-1)(3x-1)}\times 2\left(\frac{dy}{dt}\right)$	M1 f.t. – substitution
$1=\frac{2}{(2x-1)(3x-1)}$	B1 – substitute $\frac{dx}{dt}=2\frac{dy}{dt}$
$(2x-1)(3x-1)=2$	
$6x^2-5x+1-2=0$	
$6x^2-5x-1=0$	
$(x-1)(6x+1)=0$	
$x=1$ or $-\frac{1}{6}$ (rej. $\because x>\frac{1}{2}$)	A1

7 **(a)** Show that $\tan A + \cot A = 2\operatorname{cosec} 2A$. [3]

$$\begin{aligned}
\text{LHS} &= \tan A + \cot A \\
&= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} \\
&= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} \\
&= \frac{1}{\sin A \cos A} \\
&= \frac{2}{2\sin A \cos A} \\
&= \frac{2}{\sin 2A} \\
&= 2\operatorname{cosec} 2A \text{ (shown)}
\end{aligned}$$

M1 – $\frac{\sin A}{\cos A}$ or $\frac{1}{\tan A}$

M1 – for $\sin^2 A + \cos^2 A = 1$ or for $\tan^2 A + 1 = \sec^2 A$

A1 – award only if last 3 steps are shown

Alternative:

$$\begin{aligned}
\text{LHS} &= \tan A + \cot A \\
&= \tan A + \frac{1}{\tan A} \\
&= \frac{\tan^2 A + 1}{\tan A} \\
&= \frac{\sec^2 A}{\tan A} \\
&= \frac{\frac{1}{\cos^2 A}}{\frac{\sin A}{\cos A}} \\
&= \frac{1}{\sin A \cos A} \\
&= \frac{2}{2\sin A \cos A} \\
&= 2\operatorname{cosec} 2A \text{ (shown)}
\end{aligned}$$

(b) Hence, solve the equation $\frac{1}{\tan A + \cot A} = \frac{1}{4}$ for $0 \le A \le 2\pi$. [3]

$$\begin{aligned}
\frac{1}{\tan A + \cot A} &= \frac{1}{4} \\
\tan A + \cot A &= 4 \\
2\operatorname{cosec} 2A &= 4 \\
\operatorname{cosec} 2A &= 2 \\
\sin 2A &= \frac{1}{2} \\
\text{basic } \angle &= \sin^{-1}\left(\frac{1}{2}\right) \\
&= \frac{\pi}{6} \\
2A &= \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \\
A &= \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}
\end{aligned}$$

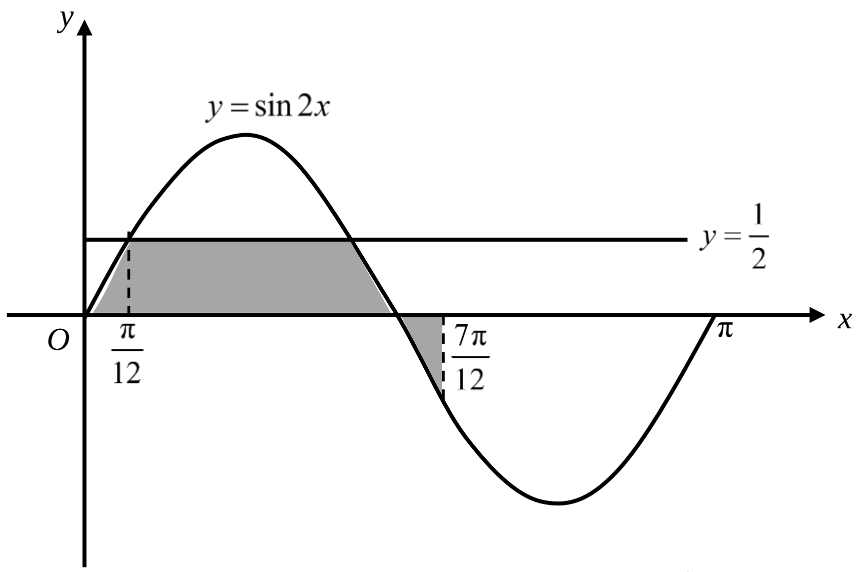
B1 – substitution of **(a)**

M1 – for finding $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

A1

(c) The diagram shows, for $0 \le x \le \pi$, the curve $y = \sin 2x$ and the line $y = \frac{1}{2}$.

Showing all your working, find the area of the shaded region. [5]



Area bounded by curve, x-axis and lines $x = 0$ and $x = \frac{\pi}{12}$

$= \int_0^{\frac{\pi}{12}} \sin 2x \, dx$

$= \left[\frac{-\cos 2x}{2} \right]_0^{\frac{\pi}{12}}$ — M1 – integrate sin $2x$

$= \left[-\frac{\cos \frac{\pi}{6}}{2} \right] - \left[-\frac{\cos 0}{2} \right]$ — DM1 – evaluate with either limits 0 to $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ to $\frac{\pi}{2}$ or $\frac{\pi}{2}$ to $\frac{7\pi}{12}$

$= \left(-\frac{\sqrt{3}}{4} + \frac{1}{2} \right)$ units^2 or 0.066987... units^2

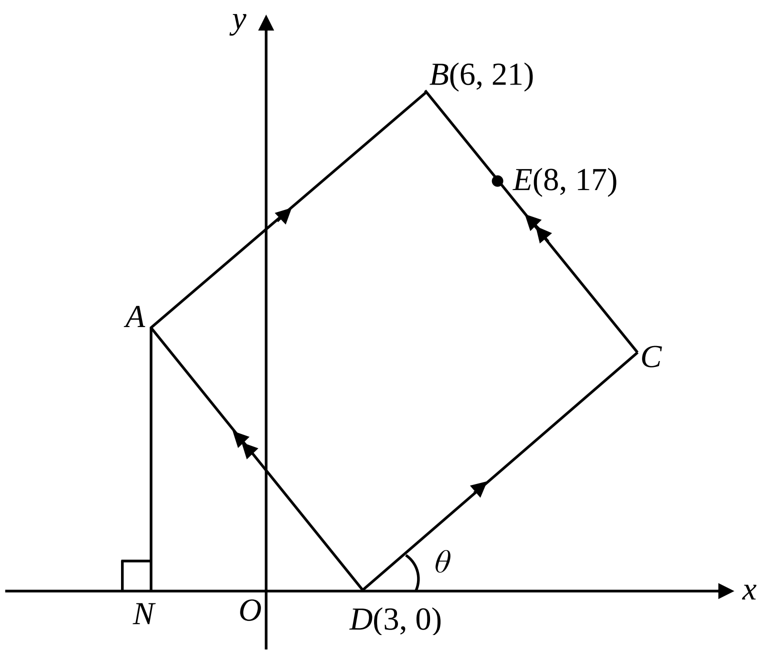
Area of rectangle $= \left(\frac{5\pi}{12} - \frac{\pi}{12} \right) \times \frac{1}{2}$ — M1 – area of rectangle

$= \frac{\pi}{6}$ units^2

Area of shaded region $= 3 \times \left(-\frac{\sqrt{3}}{4} + \frac{1}{2} \right) + \frac{\pi}{6}$ — M1 f.t. – area under curve × 3

$= 0.725$ units^2 (3 sig.fig) — A1

8



The diagram shows a parallelogram with vertices A, $B(6, 21)$, C and $D(3, 0)$. The point $E(8, 17)$ lies on BC. The line CD makes an angle θ with the positive x-axis such that $\tan\theta = 1$. A line is drawn, parallel to the y-axis, from A to meet the x-axis at N.

(a) Show that the coordinates of A are $(-3, 12)$. [5]

$m_{AB} = m_{CD}$ $= 1$	B1 – gradient = tan θ
Eq. of AB: $y - 21 = (1)(x-6)$ $y = x + 15$ ...(1)	
$m_{AD} = m_{BE}$ $= \dfrac{17-21}{8-6}$ $= -2$	B1 – gradient formula
Eq. of AD: $y - 0 = -2(x-3)$ $y = -2x + 6$ ...(2)	M1 f.t. – finding equation of AD or AB
Sub. (1) into (2): $x + 15 = -2x + 6$ $3x = -9$ $x = -3$ sub. into (1): $y = 12$	M1 f.t. – substitution
$\therefore$ $A(-3, 12)$ (shown)	A1 (A.G.)

Alternative:

Eq. of CD: $y-0=(1)(x-3)$ $y=x-3 \quad ...(1)$	
$m_{BC}=m_{BE}$ $=\dfrac{17-21}{8-6}$ $=-2$	B1 – gradient formula
Eq. of BC: $y-21=-2(x-6)$ $y=-2x+33 \quad ...(2)$	M1 f.t. – finding equation of BC
Sub. (1) into (2): $x-3=-2x+33$ $3x=36$ $x=12$ sub. into (1): $y=9$ $\therefore$ ***C*(12, 9)**	M1 f.t. – substitution
Midpoint of AC = Midpoint of BD $\left(\dfrac{x+12}{2}, \dfrac{y+9}{2}\right)=\left(\dfrac{3+6}{2}, \dfrac{0+21}{2}\right)$ $\left(\dfrac{x+12}{2}, \dfrac{y+9}{2}\right)=\left(\dfrac{9}{2}, \dfrac{21}{2}\right)$	M1 f.t. – equating midpoints
Comparing, $\dfrac{x+12}{2}=\dfrac{9}{2}$ and $\dfrac{y+9}{2}=\dfrac{21}{2}$ $x+12=9$ and $y+9=21$ $x=-3$ and $y=12$ $\therefore$ ***A*(−3, 12) (shown)**	A1 (A.G.)

(b) Hence, find the area of parallelogram *ABCD*. [2]

area of parallelogram $ABCD = 2\times$area of triangle ABD	
$= 2\times\dfrac{1}{2}\begin{vmatrix} -3 & 3 & 6 & -3 \\ 12 & 0 & 21 & 12 \end{vmatrix}$	M1 – allow clockwise direction
$= 2\times\dfrac{1}{2}\left[(63+72)-(-63+36)\right]$	
$=135+27$	
$=162 \text{ units}^2$	A1
<u>Alternative:</u> *If students have found the coordinates of C in part (a),*	
area of parallelogram $ABCD = \dfrac{1}{2}\begin{vmatrix} 3 & 12 & 6 & -3 & 3 \\ 0 & 9 & 21 & 12 & 0 \end{vmatrix}$	M1 f.t. – allow clockwise direction
$=\dfrac{1}{2}\left[(27+252+72)-(36-63+54)\right]$	
$=\dfrac{1}{2}(351-27)$	
$=162 \text{ units}^2$	A1

(c) A point F with y-coordinate of 5 lies on the line CD.
Explain why $AEFN$ is a parallelogram. [5]

Eq. of CD: $y-0=(1)(x-3)$

$y=x-3$ ...(3) — B1

Let coordinates of F be $(x, 5)$.

Sub. $(8, y)$ into (3): — M1 f.t. – substitution

$y=8-3$

$=5$

$\therefore$ $F(8, 5)$

Coordinates of $N = (-3, 0)$

length of $AN = \sqrt{(-3+3)^2+(12-0)^2}$

$=12$ units — M1 f.t. – finding either lengths

length of $EF = \sqrt{(8-8)^2+(17-5)^2}$

$=12$ units

Since F lies directly below E, the line EF is a vertical line. Thus, the lines EF and AN are parallel. — A1 – vertical lines are parallel

And that the lengths of AN and EF are equal (i.e., 12 units),

they form a pair of parallel and equal opposite sides. Thus, $AEFN$ is a parallelogram. — A1 – conclusion

Alternative 1:

gradient of $AE = \dfrac{17-12}{8-(-3)}$ $= \dfrac{5}{11}$ gradient of $FN = \dfrac{5-0}{8-(-3)}$ $= \dfrac{5}{11}$ — M1 f.t. – finding either gradients

Since gradients of lines AE and FN are the same, the lines AE and FN are parallel.

Since F lies directly below E, the line EF is a vertical line. Thus, the lines EF and AN are parallel. — A1 – vertical lines are parallel

Since there are 2 pairs of parallel lines, $AEFN$ is a parallelogram. — A1 – conclusion

Alternative 2:

Midpoint of $AF = \left(\frac{-3+8}{2}, \frac{12+5}{2}\right)$ $= \left(\frac{5}{2}, \frac{17}{2}\right)$	M1 f.t. – midpoint of *AF*
Midpoint of $NE = \left(\frac{-3+8}{2}, \frac{0+17}{2}\right)$ $= \left(\frac{5}{2}, \frac{17}{2}\right)$	M1 f.t. – midpoint of *AE*
Since the diagonals intersect at the same point, *AEFN* is a parallelogram.	A1 – conclusion

9 **(a)** Differentiate $x\ln x$ with respect to x. [2]

$$\frac{d}{dx}(x\ln x) = x\left(\frac{1}{x}\right) + (1)\ln x$$
$$= 1 + \ln x$$

B1, B1 – for each term

(b) A curve $y = f(x)$ is such that $\frac{d^2y}{dx^2} = 24x^2 + \frac{16}{x}$, where $x > 0$. The line $y = 24x - 40$ is parallel to the tangent of the curve at $P(1, -16)$.

By using the result found in part **(a)**, find the equation of the curve. [6]

$$\frac{dy}{dx} = \int 24x^2 + \frac{16}{x}\, dx$$
$$= \frac{24x^3}{3} + 16\ln x + c$$
$$= 8x^3 + 16\ln x + c, \text{ where } c \text{ is a constant.}$$

B1, B1 – for each integral

When $x = 1$, $\frac{dy}{dx} = 24$.

$$24 = 8(1)^3 + 16\ln(1) + c$$
$$c = 16$$

M1 f.t. – finding c

$$\therefore \frac{dy}{dx} = 8x^3 + 16\ln x + 16$$

$y = \int 8x^3 + 16\ln x + 16\, dx$ **or** $y = \int 8x^3 + 16(1 + \ln x)\, dx$

$$= \frac{8x^4}{4} + 16\left(x\ln x - \int 1\, dx\right) + 16x + c_1$$
$$= \frac{8x^4}{4} + 16(x\ln x - x + c_2) + 16x + c_1$$
$$= 2x^4 + 16x\ln x - 16x + 16x + c_3$$
$$2x^4 + 16x\ln x + c_3, \text{ where } c_1, c_2 \text{ and } c_3 \text{ are constants.}$$

M1 f.t. – reverse

When $x = 1$, $y = -16$.

$$-16 = 2(1)^4 + 16(1)\ln(1) + c_3$$
$$-16 = 2 + c_3$$
$$c_3 = -18$$

M1 f.t. – finding c_3

$$\therefore y = 2x^4 + 16x\ln x - 18$$

A1

Continuation of working space for question **9(b)**.

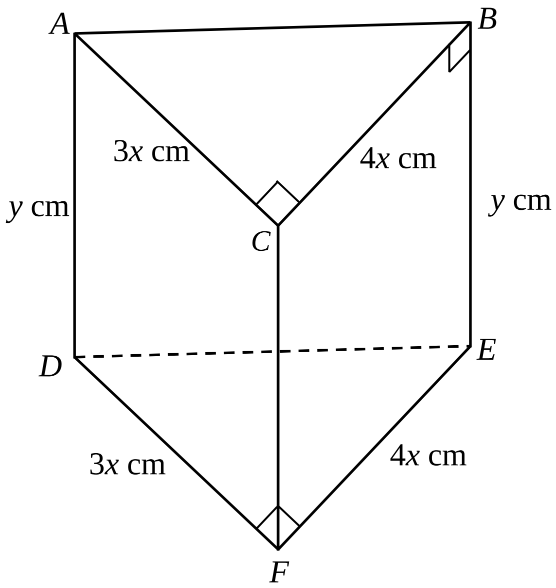
(c) Explain why the condition $x > 0$ is necessary. [1]

ln *x* is defined for $x > 0$. B1

or

ln *x* is undefined for $x < 0$.

10



The diagram shows a solid prism with right-angled triangular ends that are perpendicular to the parallel sides *AD*, *BE* and *CF*, which are each y cm in length. The right-angled triangular ends have sides *AC* and *DF*, which are $3x$ cm, and sides *BC* and *EF*, which are $4x$ cm.

Given that the volume of the prism is 1200 cm^3,

(a) find an expression for y in terms of x, [2]

$\text{Vol. of prism} = \frac{1}{2}(3x)(4x)y$	B1 – volume formula
$1200 = 6x^2y$	
$y = \frac{200}{x^2}$	B1

(b) show that the total surface area of the prism, S cm^2, is given by $S = 12x^2 + \frac{2400}{x}$. [3]

$AB = \sqrt{(3x)^2 + (4x)^2}$ $= \sqrt{25x^2}$ $= 5x$ cm	B1 – finding length of AB
Total surface area $= 2 \times \frac{1}{2}(3x)(4x) + 3xy + 4xy + 5xy$ $= 12x^2 + 12xy$ $= 12x^2 + 12x\left(\frac{200}{x^2}\right)$	M1 – at least 3 correct surfaces
$= 12x^2 + \frac{2400}{x}$ (shown)	A1 (A.G.)

(c) Given that x can vary, find the value of x for which the total surface area of the prism is a stationary value. [3]

$\frac{\mathrm{d}S}{\mathrm{d}x} = 24x - \frac{2400}{x^2}$	B1 – differentiation
For S to be stationary, $\frac{\mathrm{d}S}{\mathrm{d}x} = 0$	
$24x - \frac{2400}{x^2} = 0$ $24x = \frac{2400}{x^2}$ $x^3 = 100$ $x = \sqrt[3]{100}$	M1 f.t.
$= 4.64$ (3 sig. fig.)	A1

(d) Explain why this value of x gives the smallest surface area possible. [2]

$\dfrac{d^2S}{dx^2} = 24 + \dfrac{4800}{x^3}$	M1 f.t. – differentiation or 1st derivative test
When $x = \sqrt[3]{100}$, $\dfrac{d^2S}{dx^2} = 24 + \dfrac{4800}{\left(\sqrt[3]{100}\right)^3}$ $= 72 \quad > 0 \quad \because$ minimum	
Since $\dfrac{d^2S}{dx^2} > 0$, the surface area is the smallest when x = 4.64.	A1 – conclude '> 0' (must show '72')