1 Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_1^9 \frac{1}{x\left(2\sqrt{x}-1\right)}\,dx$. [4]

2 **(i)** Given that a is a positive constant, sketch the curve with equation $y = \frac{x+a}{x-a}$. State the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]

(ii) **(a)** Solve the inequality $\left|\frac{x+a}{x-a}\right| > 1$. [2]

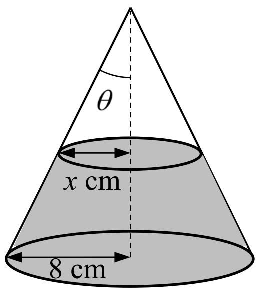
(b) Solve the inequality $\frac{|x|+a}{|x|-a} > 1$. [1]

3 **(i)** By considering $\frac{1}{(r-2)!} - \frac{1}{r!}$ for $r \geq 2$, find an expression for $\sum_{r=1}^{n} \frac{r^2 - r - 1}{r!}$ in terms of n. [3]

(ii) Give a reason why the series $\sum_{r=1}^{\infty} \frac{r^2 - r - 1}{r!}$ converges and write down its value. [2]

(iii) Show that $\sum_{r=1}^{n} \frac{r-2}{(r-1)!} < 1$ for all values of $n \geq 1$. [2]

4



Water is poured at a rate of 8 cm^3 per second into a conical container with base radius 8 cm. The semi-vertical angle of the cone is θ, where $\tan\theta = 0.4$. At time t seconds after the start, the radius of the water surface is x cm (see diagram). Find the rate of increase of the depth of water when the depth is 10 cm.

[The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.] [7]

5 With reference to the origin O, three points A, B and C have non-zero position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

(i) Show that $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$. [2]

(ii) If $\angle AOB = 180^\circ$, describe the geometrical relationship between the points A, B and C. [1]

It is given instead that $0^\circ < \angle AOB < 90^\circ$ and $\angle BOC = 2\angle AOB$.

(iii) By considering the magnitudes of the vectors on both sides of the equation in part **(i)**, show that $(\mathbf{a} + 2\mathbf{c}).(\mathbf{a} - 2\mathbf{c}) < 0$. [4]

The points A, B and C lie in the plane p which has a normal parallel to the unit vector $\mathbf{n}$.

(iv) Given that p has equation $\mathbf{r}.\mathbf{n} = k$, state the value of k. Justify your answer. [1]

6 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = Au_n + Bn + C$, where A, B and C are constants, $A \neq 0$ and $n \geqslant 1$. It is given that $u_1 = 4$.

(a) **(i)** If $A = 1$ and $B = 0$, find the value of $\sum_{r=11}^{30} u_r$ in terms of C. [3]

(ii) If instead the sequence is a geometric progression, state the values of B and C, and find the inequality satisfied by A such that $u_{20} > 2000$. [3]

(b) Given instead that $u_2 = 16$, $u_3 = 70$ and $u_4 = 334$, find u_5. [3]

7 The function f is defined by $\mathrm{f} : x \mapsto \dfrac{1}{1+x^2}, x \in \mathbb{R}, 0 < x \leq k$ where k is a real constant.

(i) Show that f has an inverse. [2]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

It is further given that $k < \dfrac{1}{1+k^2}$.

(iii) Sketch the graph of $y = \mathrm{f}^{-1}(x)$. Your sketch should indicate the position of the graph in relation to the line $y = x$. [3]

(iv) Deduce the number of solutions of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [1]

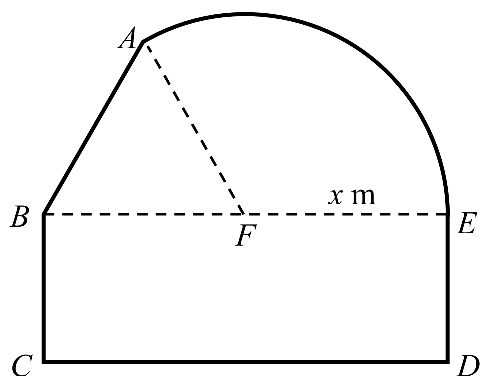
8 **(a)** The complex number w is given by $w = re^{i\theta}$, where $r > 0$ and $0 \leq \theta \leq \dfrac{\pi}{2}$.

(i) Given that $z = (1 - i\sqrt{3})w$, find $|z|$ in terms of r and $\arg(z)$ in terms of θ. [2]

(ii) Given that $\left(\dfrac{z^2}{z^*}\right)^2 = 36i$, find r and all possible values of θ. [5]

(b) The equation $z^3 + pz^2 + 17z - 13 = 0$, where p is real, has a root $z = 2 + qi$ where q is a real positive constant. Find the values of p and q, showing your working. [5]

9



[Turn Over

The diagram above shows a plan view of a livestock enclosure, $ABCDEA$, consisting of a rectangle $BCDE$ joined to an equilateral triangle BFA and a sector of a circle with radius x metres and centre F. The points B, F and E lie on a straight line with $FE = x$ metres.

(i) Find the exact area of sector FEA, giving your answer in terms of x and in a simplified form. [2]

To ensure sufficient living space is given to the livestock, it is desired to have the area of the enclosure as 1000 m^2. At the same time, to save on the cost required for fencing the enclosure, the perimeter of the enclosure, P metres, should be made as small as possible.

(ii) Show that

$$P = \frac{1000}{x} + \frac{x}{12}\left(4\pi + 36 - 3\sqrt{3}\right).$$ [4]

(iii) Use calculus to find the minimum value of P, giving your answer to one decimal place. Justify, by further differentiation, that the value of P you have found is a minimum. [6]

10 **(i)** Find $\int \frac{1}{2 + x - x^2}\, dx$, given that $0 \le x \le \frac{1}{2}$. [3]

(ii) The mass, x grams, of a substance, X, present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{dx}{dt} = k(2 + x - x^2)$$

where $0 \le x \le \frac{1}{2}$ and k is a constant. It is given that $x = \frac{1}{2}$ and $\frac{dx}{dt} = -\frac{1}{9}$ when $t = 0$

(a) Find t in terms of x. [3]

(b) Find the exact time for there to be none of the substance X present in the chemical reaction. [1]

The mass, y grams, of another substance, Y, present in the chemical reaction at time t minutes satisfies the differential equation $\frac{d^2y}{dt^2} = \frac{1}{t+1}$. It is given that $y = 0$ and $\frac{dy}{dt} = 0$ when $t = 0$

(c) Find y in terms of t. [5]

11 A curve C has parametric equations

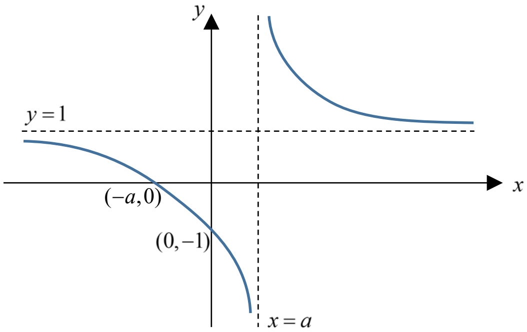
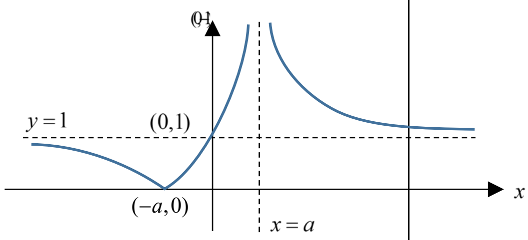
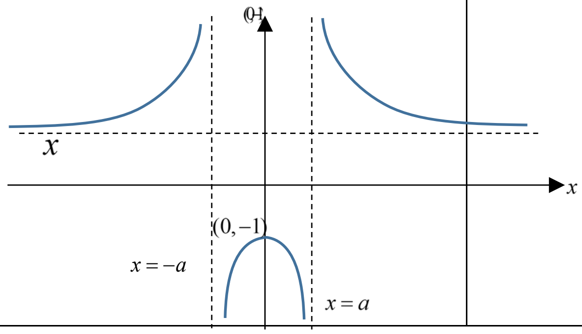
$$x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t,$$

where $0 \le t \le \frac{\pi}{2}$. The line l is the normal to C at the point where $t = \frac{\pi}{3}$.

(i) Find the equation of l. [3]

(ii) On the same diagram, sketch C and l. [2]

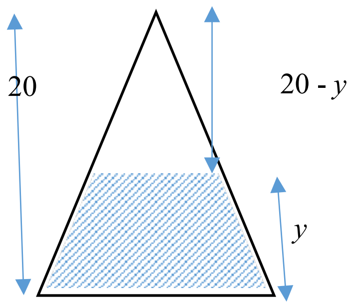
(iii) Find the exact area of the region bounded by C, l and the y-axis. [9]

1	$u=\sqrt{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1}{2\sqrt{x}}$ when $x=9,\ u=3$ when $x=1,\ u=1$ $\displaystyle\int_1^9 \frac{1}{x\left(2\sqrt{x}-1\right)}\,\mathrm{d}x=\int_1^3 \frac{1}{u^2\left(2u-1\right)}\,2u\ \mathrm{d}u$ $\displaystyle=\int_1^3 \frac{2}{u\left(2u-1\right)}\ \mathrm{d}u$ $\displaystyle=\int_1^3 \frac{-2}{u}+\frac{4}{2u-1}\ \mathrm{d}u$ $=\left[-2\ln\lvert u\rvert+2\ln\lvert 2u-1\rvert\right]_1^3$ $=-2\ln 3+2\ln 5$ $=2\ln\left(\dfrac{5}{3}\right)$	
2(i)		
2(ii) (a)	From the graph, $0<x<a$ or $x>a$ 	
2(ii) (b)	From the graph, $x<-a$ or $x>a$ 	

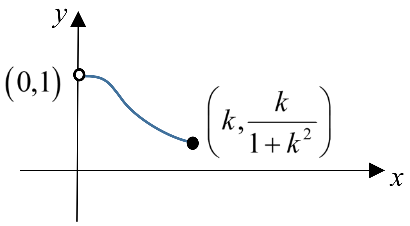
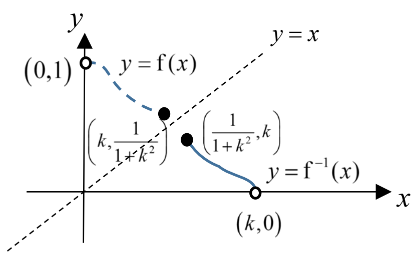
3(i)	$\frac{1}{(r-2)!}-\frac{1}{r!}=\frac{r(r-1)-1}{r!}=\frac{r^2-r-1}{r!}$ $\sum_{r=1}^{n}\frac{r^2-r-1}{r!}=\sum_{r=2}^{n}\left(\frac{1}{(r-2)!}-\frac{1}{r!}\right)+\frac{1-1-1}{1!}$ $=\left(\frac{1}{0!}-\cancel{\frac{1}{2!}}\right.$ $+\frac{1}{1!}-\cancel{\frac{1}{3!}}$ $+\cancel{\frac{1}{2!}}-\cancel{\frac{1}{4!}}$ $+\ldots$ $+\cancel{\frac{1}{(n-4)!}}-\cancel{\frac{1}{(n-2)!}}$ $+\cancel{\frac{1}{(n-3)!}}-\frac{1}{(n-1)!}$ $\left.+\cancel{\frac{1}{(n-2)!}}-\frac{1}{n!}\right)\quad -1$ $=1+1-\frac{1}{(n-1)!}-\frac{1}{n!}-1$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}$	
3(ii)	As $n\to\infty$, $-\frac{1}{(n-1)!}\to 0$ and $-\frac{1}{n!}\to 0$ $\therefore \sum_{r=1}^{n}\frac{r^2-r-1}{r!}\to 1$, a finite value $\therefore \sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}$ converges sum to infinity $=\sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}=1$	
3(iii)	$\sum_{r=1}^{n}\frac{r-2}{(r-1)!}=\sum_{r=1}^{n}\frac{r^2-2r}{r!}=\sum_{r=1}^{n}\frac{r^2-r-r}{r!}$ $<\sum_{r=1}^{n}\frac{r^2-r-1}{r!}\quad\left(\because \frac{r}{r!}>\frac{1}{r!}\text{ for }r>1\right)$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}<1$	

Note to tutors:
In recent years, A level phrasing of 3(ii) is "Give a reason why the series converges and write down the value of the sum to infinity". With such phrasing, Cambridge expects students to write "sum to infinity = ___" (See report below).
N07/II/2

(iii) Adequate answers to this part were rare. Many candidates seemed to be unable to think of convergence of series that were not geometric; answers such as 'it converges because $|r| < 1$' were common. Nor is it correct to say that it converges because the terms decrease, as is disproved by the harmonic series. The correct statements are that $\frac{1}{(N+1)^2}$ tends to 0 (*not* '$\frac{1}{\infty}$')

4	Let the volume of water in the cone be V cm^3 at time t seconds, when the depth is y cm.  $\tan\theta = \frac{8}{h} \Rightarrow h = 20$ $V = \frac{1}{3}\pi\left(8^2\right)(20) - \frac{1}{3}\pi\left(x^2\right)(20-y),$ Since $\tan\theta = \frac{x}{20-y}$, $V = \frac{1280\pi}{3} - \frac{1}{3}\pi\left[0.4(20-y)\right]^2(20-y)$ $= \frac{1280\pi}{3} - \frac{0.16}{3}\pi(20-y)^3$ $\frac{\mathrm{d}}{\mathrm{d}t}$: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.16\pi(20-y)^2\frac{\mathrm{d}y}{\mathrm{d}t}$ When $y = 10$, $\frac{\mathrm{d}V}{\mathrm{d}t} = 8$, $8 = 0.16\pi(10)^2\frac{\mathrm{d}y}{\mathrm{d}t}$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 0.159$ Depth of water increases at 0.159 cm per second.	
5(i)	$\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ $(\mathbf{a}+\mathbf{b}+\mathbf{c})\times\mathbf{b}=\mathbf{0}\times\mathbf{b}$ $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{b}+\mathbf{c}\times\mathbf{b}=\mathbf{0}$ $\mathbf{a}\times\mathbf{b}+\mathbf{0}+\mathbf{c}\times\mathbf{b}=\mathbf{0}$ $\mathbf{a}\times\mathbf{b}=-(\mathbf{c}\times\mathbf{b})$ $=\mathbf{b}\times\mathbf{c}$	
5(ii)	A, B and C are collinear (lie on a straight line).	

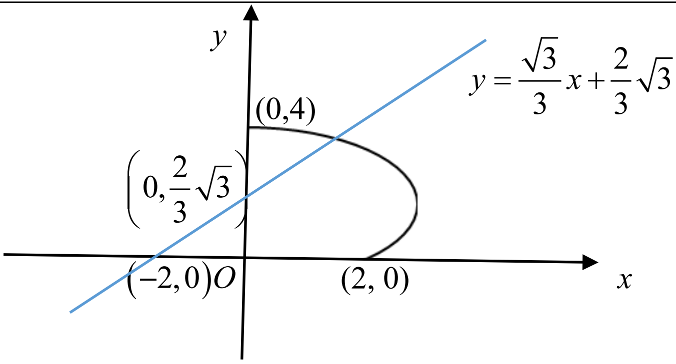
5(iii)	$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$ (from part (i)) $\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\angle AOB=\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert\sin\angle BOC$ $\lvert\mathbf{a}\rvert\sin\angle AOB=\lvert\mathbf{c}\rvert\sin(2\angle AOB)$ $\quad(\text{since } \lvert\mathbf{b}\rvert\neq 0)$ $=2\lvert\mathbf{c}\rvert\sin\angle AOB\cos\angle AOB$ $\lvert\mathbf{a}\rvert=2\lvert\mathbf{c}\rvert\cos\angle AOB$ $(\text{since } \sin\angle AOB\neq 0)$ $0=\lvert\mathbf{a}\rvert^2-4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB$ $\lvert\mathbf{a}\rvert^2=4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB$ $(\mathbf{a}+2\mathbf{c}).(\mathbf{a}-2\mathbf{c})$ $=\mathbf{a}.\mathbf{a}+(2\mathbf{c}).\mathbf{a}-\mathbf{a}.(2\mathbf{c})+(2\mathbf{c}).(2\mathbf{c})$ $=\lvert\mathbf{a}\rvert^2-4\lvert\mathbf{c}\rvert^2$ $=4\lvert\mathbf{c}\rvert^2\cos^2\angle AOB-4\lvert\mathbf{c}\rvert^2$ $=4\lvert\mathbf{c}\rvert^2(\cos^2\angle AOB-1)$ $=4\lvert\mathbf{c}\rvert^2(-\sin^2\angle AOB)<0$ $(\because \lvert\mathbf{c}\rvert^2\geq 0 \text{ and} \sin^2\angle AOB>0$	
5(iv)	$k=0$ since the origin O is in the plane p.	
6(a) (i)	$A=1, B=0 \Rightarrow u_{n+1}=u_n+C \Rightarrow$ AP $\sum_{r=11}^{30}u_r=\frac{20}{2}(u_{11}+u_{30})$ $=10(4+10C+4+29C)$ $=80+390C$	
(ii)	GP $\Rightarrow u_{n+1}=Au_n$ for all values of n $\therefore B=0, C=0$ $U_{20}>2000 \Rightarrow 4(A)^{19}>2000$ $\Rightarrow A>500^{\frac{1}{19}}$ $\Rightarrow A>1.39$	
(b)	$u_{n+1}=Au_n+Bn+C$ $n=1:\quad 16=4A+B+C$ $n=2:\quad 70=16A+2B+C$ $n=3:\quad 334=70A+3B+C$ From GC, $A=5,\ B=-6, C=2$ $\therefore u_5=5u_4-6(4)+2=1648$	

7(i)	 **Every** horizontal line $y = k$ cuts the graph of $y = \mathrm{f}(x)$ at most **once**. Hence **f is one-one** and the inverse of f exists.	
7 (ii)	$y=\mathrm{f}(x)=\dfrac{1}{1+x^2}$ $\dfrac{1}{y}=1+x^2$ $x^2=\dfrac{1}{y}-1$ $x=\sqrt{\dfrac{1}{y}-1}$ $\because$ reject $-\sqrt{\dfrac{1}{y}-1}$ since $x>0$ $\mathrm{f}^{-1}(x)=\sqrt{\dfrac{1}{x}-1}$ domain of f^{-1} = $\left[\dfrac{1}{1+k^2},1\right)$	
7 (iii)		
7(iv)	Since $y=\mathrm{f}^{-1}(x)$ do not intersect the line $y = x$, it will not intersect $y=\mathrm{f}(x)$ too, hence no solutions for $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. **Alternative:** Since the domains of f and f^{-1} are different, there is no solution for $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$	

8(a) (i)	$\arg(z)=\arg\left(1-\mathrm{i}\sqrt{3}\right)+\arg(w)$ $=-\dfrac{\pi}{3}+\theta$ $\lvert z\rvert=\left\lvert 1-\mathrm{i}\sqrt{3}\right\rvert \lvert w\rvert$ $=2r$
8(a) (ii)	$\arg\left(\dfrac{z^2}{z^*}\right)^2=4\arg(z)-2\arg\left(z^*\right)$ $=4\arg(z)+2\arg\left(z\right)$ $=6\theta+2k\pi \quad k\in\mathbb{Z}$ $\arg\left(36\mathrm{i}\right)=\dfrac{\pi}{2}$ $\lvert 36\mathrm{i}\rvert=36$ $\left\lvert\left(\dfrac{z^2}{z^*}\right)^2\right\rvert=\dfrac{\lvert z\rvert^4}{\lvert z\rvert^2}$ $=4r^2$ Hence, $4r^2=36\Rightarrow r=3$. $6\theta+2k\pi=\dfrac{\pi}{2}$ $\theta=\dfrac{\pi-4k\pi}{12}$ For $0\le\theta\le\dfrac{\pi}{2}$, we have $k=-1,\ 0$, giving us $\theta=\dfrac{5\pi}{12},\ \dfrac{\pi}{12}$
8(b)	Since p is a real value, by conjugate root theorem, $2-q\mathrm{i}$ is also a root. $\left(z-\left(2+q\mathrm{i}\right)\right)\left(z-\left(2-q\mathrm{i}\right)\right)=z^2-4z+4+q^2$ Let the third root be v. We know that v is real. $z^3+pz^2+17z-13=\left(z^2-4z+4+q^2\right)\left(z-v\right)$ Coefficient of z^0: $-(4+q^2)v=-13$ ----(1) Coefficient of z: $4v+4+q^2=17\Rightarrow q^2=13-4v$ ---(2) Sub (2) into (1), we have $\left(17-4v\right)v=13$ $4v^2-17v+13=0$ $(4v-13)(v-1)=0$ $v=\dfrac{13}{4}$ or $v=1$ However, if $v=\dfrac{13}{4}$, sub into (2), gives us $q=0$. Hence, $v\ne\dfrac{13}{4}$ and $v=1$. $\therefore z^3+pz^2+17z-13=\left(z^2-4z+4+q^2\right)\left(z-1\right)$ $z^3+pz^2+17z-13=z^3-5z^2+(q^2+4)z-4-q^2$ Hence, $p=-5$ and $q=3$.
9 (i)	Angle $AFE=180^\circ-60^\circ=120^\circ$ Area of sector FEA = $\dfrac{120}{360}\pi x^2=\dfrac{1}{3}\pi x^2$

9 (ii)	total area = $\frac{1}{2}x^2 \sin 60^\circ + BC(2x) + \frac{1}{3}\pi x^2$ $2x(BC) = 1000 - \frac{\sqrt{3}}{4}x^2 - \frac{1}{3}\pi x^2$ $BC = \frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x$ $P = x + 2\left(\frac{500}{x} - \frac{\sqrt{3}}{8}x - \frac{\pi}{6}x\right) + 2x + \frac{1}{3}(2\pi x)$ $= \frac{1000}{x} + \frac{x}{12}\left(4\pi + 36 - 3\sqrt{3}\right)$
9 (iii)	$\frac{dP}{dx} = \frac{-1000}{x^2} + \frac{1}{12}\left(36 - 3\sqrt{3} + 4\pi\right)$ $\frac{dP}{dx} = 0 \Rightarrow \frac{1000}{x^2} = \frac{1}{12}\left(36 - 3\sqrt{3} + 4\pi\right)$ $x = 16.634$ $P = 120.2$ $\frac{d^2P}{dx^2} = \frac{2000}{x^3}$ When $x = 16.634$, $\frac{d^2P}{dx^2} = \frac{2000}{16.634^3} > 0$ $\therefore$ minimum $P = 120.2$
10(i)	$\int \frac{1}{2 + x - x^2}\, dx$ $= \int \frac{1}{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2}\, dx$ $= \frac{1}{2\left(\frac{3}{2}\right)} \ln \left\lvert \frac{\frac{3}{2} + \left(x - \frac{1}{2}\right)}{\frac{3}{2} - \left(x - \frac{1}{2}\right)} \right\rvert + C$ $= \frac{1}{3}\ln\left(\frac{1+x}{2-x}\right) + C \quad \because x \le \frac{1}{2}$

10(ii) (a)	Given $x=\frac{1}{2},\ \frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{4}$ $\therefore -\frac{1}{9}=k\left(2+\frac{1}{2}-\frac{1}{4}\right)$ $\Rightarrow \frac{9}{4}k=-\frac{1}{4}\Rightarrow k=-\frac{1}{9}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{9}(2+x-x^2)$ $\int \frac{1}{2+x-x^2}\,\mathrm{d}x=\int -\frac{1}{9}\,\mathrm{d}t$ $-\frac{1}{9}t=\frac{1}{3}\ln\left(\frac{1+x}{2-x}\right)+C$ $t=-3\ln\left(\frac{1+x}{2-x}\right)+C$ When $t=0,\ x=\frac{1}{2},\ \therefore 0=-3\ln 1+C\Rightarrow C=0$ $t=-3\ln\left(\frac{1+x}{2-x}\right)$
10(b) (ii)	At $x=0,\ t=-3\ln\left(\frac{1}{2}\right)=3\ln 2$
10(b) (iv)	$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\frac{1}{t+1}$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\ln\lvert t+1\rvert+C$ $=\ln(t+1)+C \quad \because t+1\geq 1>0$ When $t=0,\ \frac{\mathrm{d}y}{\mathrm{d}t}=0\Rightarrow\ 0=\ln(0+1)+C\Rightarrow C=0$ $\frac{\mathrm{d}y}{\mathrm{d}t}=\ln(t+1)$ $y=\int \ln(t+1)\ \mathrm{d}t$ $=t\ln(t+1)-\int \frac{t}{t+1}\ \mathrm{d}t$ $=t\ln(t+1)-\int 1-\frac{1}{t+1}\ \mathrm{d}t$ $=t\ln(t+1)-t+\ln(t+1)+D$ When $t=0,\ y=1,\ D=0$ Hence, $y=(t+1)\ln(t+1)-t$

11(i)	(i) $\frac{dy}{dx}=\frac{3\cos t-3\cos 3t}{-3\sin t+3\sin 3t}$ When $t=\frac{\pi}{3}, x=\frac{5}{2}, y=\frac{3\sqrt{3}}{2}, \frac{dy}{dx}=-\sqrt{3}$ Equation of normal: $y-\frac{3\sqrt{3}}{2}=\frac{\sqrt{3}}{3}\left(x-\frac{5}{2}\right)\Rightarrow y=\frac{\sqrt{3}}{3}x+\frac{2}{3}\sqrt{3}$
11(ii)	(ii) 

11 (iii)	(iii) Point of intersection $\left(\frac{5}{2},\frac{3\sqrt{3}}{2}\right)$ Area $=\int_0^{5/2} y\ \mathrm{d}x - \text{area of trapizum}$ $=\int_{\pi/2}^{\pi/3}(3\sin t-\sin(3t))3(-\sin t+\sin 3t)\ \mathrm{d}t-\frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{2}{3}\sqrt{3}+\frac{3}{2}\sqrt{3}\right)$ $=3\int_{\pi/2}^{\pi/3}\left(-3\sin^2 t+4\sin t\sin(3t)-\sin^2(3t)\right)\ \mathrm{d}t-\frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{13}{6}\sqrt{3}\right)$ $=3\int_{\pi/2}^{\pi/3}\left(3\left(\frac{\cos 2t-1}{2}\right)-2(\cos 4t-\cos 2t)+\frac{\cos 6t-1}{2}\right)\mathrm{d}t-\frac{65}{24}\sqrt{3}$ $=3\int_{\pi/2}^{\pi/3}\left(-2+\frac{7}{2}\cos 2t-2\cos 4t+\frac{\cos 6t}{2}\right)\mathrm{d}t-\frac{65}{24}\sqrt{3}$ $=3\left[-2t+\frac{7}{4}\sin 2t-\frac{2\sin 4t}{4}+\frac{\sin 6t}{12}\right]_{\pi/2}^{\pi/3}-\frac{65}{24}\sqrt{3}$ $=3\left[-\frac{2\pi}{3}+\frac{7}{4}\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{4}-(-\pi)\right]-\frac{65}{24}\sqrt{3}$ $=\pi+\frac{2}{3}\sqrt{3}$ or Area $=\int_{3\sqrt{3}/2}^{4} x\ \mathrm{d}y + \text{area of triangle}$ $=\int_{\pi/3}^{\pi/2}(3\cos t-\cos(3t))3(\cos t-\cos 3t)\ \mathrm{d}t+\frac{1}{2}\left(\frac{5}{2}\right)\left(\frac{3}{3}\sqrt{3}-\frac{2}{3}\sqrt{3}\right)$ $=3\int_{\pi/3}^{\pi/2}\left(3\cos^2 t-4\cos t\cos(3t)+3\cos^2(3t)\right)\mathrm{d}t+\frac{25}{24}\sqrt{3}$ $=3\int_{\pi/3}^{\pi/2}\left(3\left(\frac{\cos 2t+1}{2}\right)-2(\cos 4t+\cos 2t)+\frac{\cos 6t+1}{2}\right)\mathrm{d}t+\frac{25}{24}\sqrt{3}$ $=3\int_{\pi/3}^{\pi/2}\left(2-\frac{1}{2}\cos 2t-2\cos 4t+\frac{\cos 6t}{2}\right)\mathrm{d}t+\frac{25}{24}\sqrt{3}$ $=3\left[2t-\frac{1}{4}\sin 2t-\frac{2\sin 4t}{4}+\frac{\sin 6t}{12}\right]_{\pi/3}^{\pi/2}+\frac{25}{24}\sqrt{3}$ $=3\left[\pi-\left(\frac{2\pi}{3}-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)+\frac{\sqrt{3}}{4}\right)\right]+\frac{25}{24}\sqrt{3}$ $=\pi+\frac{2}{3}\sqrt{3}$

Section A: Pure Mathematics [44 marks]

1 The function f is defined by

$$\mathrm{f}: x \mapsto \ln(2x+1)+5, \quad x \in \mathbb{R}, x > -\frac{1}{2}.$$

(i) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\dfrac{1}{\mathrm{f}(x)}$, giving the equations of any asymptotes and the coordinates of any points where the graphs cross the axes. [6]

(ii) Describe a sequence of three transformations which would transform the curve $y=\ln(-2x)$ onto the curve $y=\mathrm{f}(x)$. [3]

2 A curve C has equation $y^2+k(x-3)^2=4$, where k is a real constant.

(a) Sketch, on separate clearly labelled diagrams, the graphs of C for both $k>0$ and $k<0$. State, in terms of k, the coordinates of any points where the curves cross the x-axis, the equations of any asymptotes and the coordinates of any points of intersection of the asymptotes and stationary points. [5]

(b) Assume now that $0<k<1$.

(i) The region S is bounded by C and the lines $x=3$ and $x=4$. Find the volume of the solid of revolution formed when S is rotated about the x-axis through 180°, giving your answer in terms of k. [3]

The curve C is stretched with scale factor $\dfrac{1}{2}$ parallel to the y-axis to form the curve D.

(ii) Without integration, state the volume of the solid of revolution formed when the region bounded by D and the lines $x=3$ and $x=4$ is rotated about the x-axis through $180°$, giving your answer in terms of k. [1]

3 **(a)** Expand $\dfrac{4+x^2}{\sqrt{4-x}}$ in ascending powers of x, up to and including the term in x^2. [3]

State the set of values of x for which the expansion is valid. [1]

(b) It is given that $y=\ln(1+\sin x)$.

(i) Show that $(1+\sin x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+(\cos x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sin x$. [2]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for y, up to and including the term in x^3. [4]

(iii) Hence deduce the Maclaurin series for $\ln\left(\dfrac{1-\sin x}{2}\right)$, up to and including the term in x^3. [2]

4 The planes p_1 and p_2 have equations $\mathbf{r}=\begin{pmatrix}0\\3\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\3\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $4x+y+3z=1$ respectively, where λ and μ are parameters. These two planes meet in the line l.

(i) Show that p_1 is perpendicular to p_2. [2]

(ii) Explain why l passes through the point $(1,0,-1)$ and find a vector equation for l. [3]

(iii) The points $A(0,3,-5)$ and $B(2,2,-3)$ lie in p_1 and p_2 respectively and the point B' is the reflection of point B in the line l. Find the exact area of triangle ABB'. [5]

A third plane p_3 has equation $\mathbf{r}.\begin{pmatrix}a\\3\\1\end{pmatrix}=-9$, where a is a constant. The point Q lies in all three planes p_1, p_2 and p_3.

(iv) Explain why Q lies on the line l. [1]

(v) Hence or otherwise, find the coordinates of Q, showing your working. [3]

Section B: Statistics [56 marks]

5 Find the number of ways in which all twelve letters of the word MISSPELLINGS can be arranged if

(i) both the I's are placed at the beginning and both the L's are placed at the end, [1]

(ii) between any two S's, there must be at least 4 other letters. [3]

A group of 8 letters is randomly selected from the letters of the word MISSPELLINGS. Find the probability that all 8 letters are distinct. [2]

6 A scientist is interested to study the behaviour of raccoons.

N boxes are placed in a room with exactly 1 of them containing food. Alex, the raccoon, will randomly open one of the boxes to see if it contains food. It will stop once it finds food, else, it will randomly open another box until it finds food. You may assume that Alex is trained such that it will not attempt to open the same box more than once. Let X be the number of boxes that Alex opens.

(i) If $N=3$, determine the probability distribution of X. [2]

(ii) State $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=\dfrac{(N-1)(N+1)}{12}$. [4]

[You may use the result $\displaystyle\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$.]

7 For events A, B and C it is given that $\mathrm{P}(A)=0.8$, $\mathrm{P}(B)=0.4$, $\mathrm{P}(C)=c$ and $\mathrm{P}(A\cup B)=0.95$.

(i) Find $\mathrm{P}(A\cap B)$ and determine with reason, if events A and B are independent. [2]

(ii) State the range of values of c which necessarily implies that events A and C are **not** mutually exclusive. [1]

It is now given that $\mathrm{P}(C)=\mathrm{P}(C\mid A)=0.45$.

(iii) Find $\mathrm{P}(A\cap C)$ and state the greatest and least possible values of $\mathrm{P}(A'\cap B\cap C)$. [3]

8 **(a)** The random variable Y has the distribution $\mathrm{B}(n,p)$ with mean 2.75. Given that $\mathrm{P}(Y<2)=0.13122$, find the value of n and the value of p. [3]

(b) On average a% of the residents in a city use the bicycle-sharing platform, ShareBike. A sample of n residents is taken and the random variable X denotes the number of residents in the sample who use ShareBike.

(i) State, in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has a binomial distribution.

(ii) Given that $n=45$ and $a=8$, find the probability that at least 9 but not more than 13 residents use ShareBike. [2]

(iii) It is given instead that $n=12$ and the modal number of residents who use ShareBike in the sample is 2. Use this information to find exactly the range of values that a can take. [4]

9 A factory supplies beans in small cans. The mass of one can of beans is denoted by X grams. A random sample of 40 cans of beans was taken and the masses are summarised as follows.

$$\sum(x-425)=-136 \quad \text{and} \quad \sum(x-425)^2=4927.5$$

(i) Calculate unbiased estimates of the population mean and variance of the mass of cans of beans. [2]

(ii) Test, at the 5% significance level, the claim that the mean mass of a can of beans is 425 grams. You should state your hypotheses and define any symbols you use. [5]

(iii) State, giving a reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

(iv) Explain, in the context of the question, the meaning of ‘at the 5% significance level’. [1]

The factory also supplies frozen corn in packets. The mass of a randomly chosen packet of frozen corn has a normal distribution with standard deviation 12 grams. The factory claims that the mean mass of the packets of frozen corn is 380 grams. However, a random sample of 15 packets of frozen corn is taken and the mean mass of the sample is found to be 375 grams.

(v) Given that a one-tail test at the α % significance level concludes that there is insufficient evidence to reject the factory’s claim, find the set of possible values of α. [3]

10 In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.

A certain bakery bakes two types of cookies; butter cookies and chocolate cookies. The masses of butter cookies have the distribution $\mathrm{N}\left(15, 0.4^2\right)$ and the masses of chocolate cookies have the distribution $\mathrm{N}\left(20, 1.2^2\right)$. The units for mass are grams.

(i) Find the probability that the mass of a randomly chosen butter cookie is more than 15.5 grams. [1]

(ii) 10 butter cookies are randomly chosen. Find the probability that at least 4 of them each has mass more than 15.5 grams. [3]

The cookies are sold by weight. Butter cookies cost \$6 per 100 grams and chocolate cookies cost \$7.50 per 100 grams.

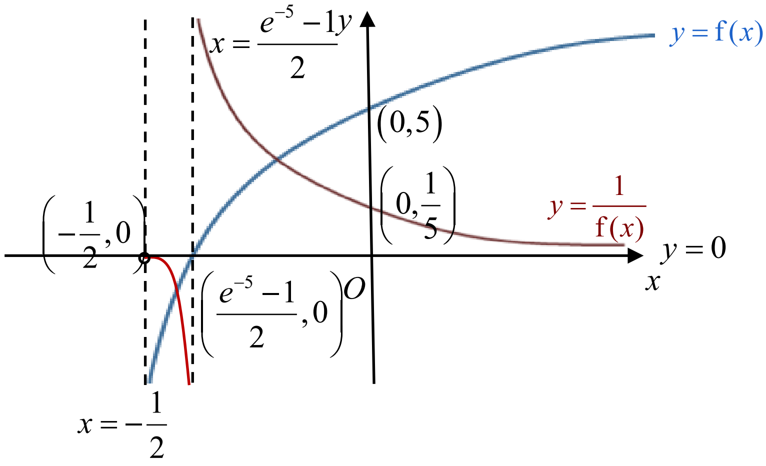
(iii) Miss Lee bought 12 butter cookies and 12 chocolate cookies for her family. Find the probability that she paid less than \$29. [4]

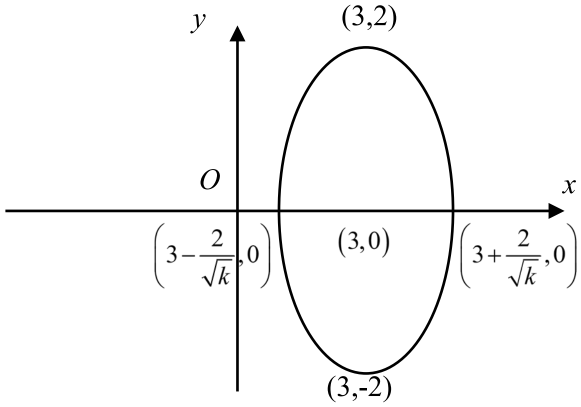
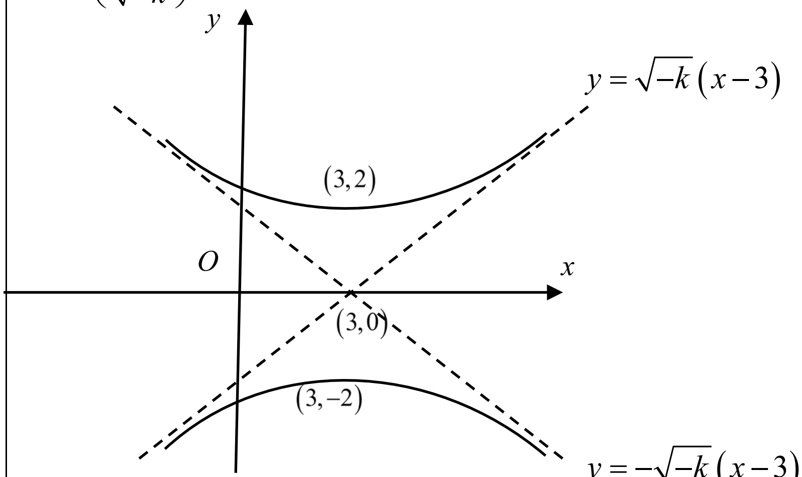
(iv) State an assumption needed for your calculations in part **(iii)**. [1]

The waiting time, T minutes, before a customer is served at the bakery has a mean of 16 minutes and a standard deviation of 9 minutes.

(v) Give a reason why a normal distribution, with this mean and standard deviation, would not give a good approximation to the distribution of T. [1]

(vi) The waiting times of n randomly chosen customers at the bakery are taken, where $n > 30$. Given that the probability that the average waiting time of these n customers is between 16 minutes and 18 minutes is more than 0.48, find the least value of n. [5]

1(i)		
1 (ii)	In the equation $y=\ln(-2x)$, replace x with $(-x)$ gives us $y=\ln(2x)$. First, reflect the graph of $y=\ln(-2x)$ in the y-axis. In the equation $y=\ln(2x)$, replace x with $x+\frac{1}{2}$ gives us $y=\ln(2x+1)$. Next, translate the graph of $y=\ln(2x)$, $\frac{1}{2}$ units in the negative x-direction. In the equation $y=\ln(2x+1)$, replace y with $y-5$ gives us $y=\ln(2x+1)+5$. Lastly, translate the graph of $y=\ln(2x+1)$, 5 units in the positive y-direction.	

2(a)	$y^2+k(x-3)^2=4$ $k>0$ $\frac{y^2}{2^2}+\frac{(x-3)^2}{\left(2/\sqrt{k}\right)^2}=1$  $k<0$ $y^2-(-k)(x-3)^2=4$ $\frac{y^2}{2^2}-\frac{(x-3)^2}{\left(\frac{2}{\sqrt{-k}}\right)^2}=1$  	

(b) (i)	Volume $= \pi \int_3^4 \left[4 - k(x-3)^2 \right] \mathrm{d}x$ $= \pi \left[4x - \frac{k}{3}(x-3)^3 \right]_3^4$ $= \pi \left(16 - \frac{k}{3} - 12 \right)$ $= \pi \left(4 - \frac{k}{3} \right)$	
(ii)	Volume $= \int_3^4 \left(\frac{1}{2} \right)^2 \left[4 - k(x-3)^2 \right] \mathrm{d}x$ $= \left(\frac{1}{2} \right)^2 \int_3^4 \left[4 - k(x-3)^2 \right] \mathrm{d}x$ $= \left(\frac{1}{2} \right)^2 \pi \left(4 - \frac{k}{3} \right)$ $= \frac{\pi}{4} \left(4 - \frac{k}{3} \right)$	

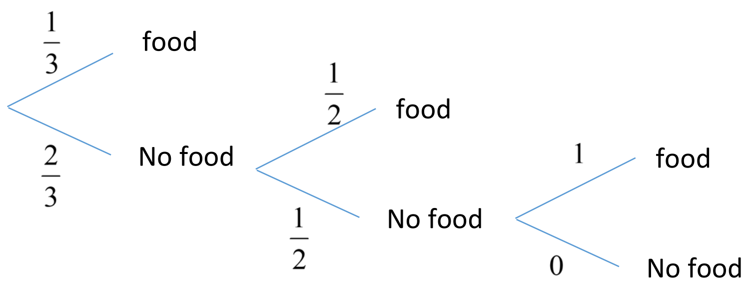
3 (a)	$\frac{4+x^2}{\sqrt{4-x}}=\left(4+x^2\right)\left(4-x\right)^{-\frac{1}{2}}$ $=4^{-\frac{1}{2}}\left(4+x^2\right)\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$ $=\frac{1}{2}\left(4+x^2\right)\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{x}{4}\right)^2+...\right)$ $=\frac{1}{2}\left(4+x^2\right)\left(1+\frac{x}{8}+\frac{3}{128}x^2+...\right)$ $=\frac{1}{2}\left(4+\frac{1}{2}x+\frac{3}{32}x^2+x^2+...\right)$ $=2+\frac{1}{4}x+\frac{35}{64}x^2+...$ expansion is valid for $\left\|-\frac{x}{4}\right\|<1$ $\Rightarrow -4<x<4$ $\therefore \left\{x\in\mathbb{R}:-4<x<4\right\}$	
3(b) (i)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos x}{1+\sin x}$ Differentiating $\left(1+\sin x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ w.r.t x, $\left(1+\sin x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\cos x\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin x$ (shown)	

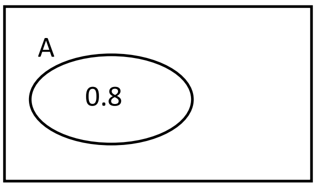
3(ii)	Differentiating w.r.t. x, $\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+\sin x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\frac{\mathrm{d}y}{\mathrm{d}x}+\cos x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\cos x$ When $x=0$, $y=\ln(1+0)=0$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos 0}{1+\sin 0}=1$ $(1+0)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+1=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1$ $(1)(-1)+(1+0)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-(0)(1)+(1)(-1)=-1$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=1$ $y=0+(1)x+\frac{(-1)}{2!}x^2+\frac{(1)}{3!}x^3+...$ $=x-\frac{1}{2}x^2+\frac{1}{6}x^3+...$	
3(iii)	$\ln\left(\frac{1-\sin x}{2}\right)=\ln(1+\sin(-x))-\ln 2$ $=-\ln 2+(-x)-\frac{1}{2}(-x)^2+\frac{1}{6}(-x)^3+...$ $=-\ln 2-x-\frac{1}{2}x^2-\frac{1}{6}x^3+...$	
4(i)	A normal to p_1 is $\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}3\\-3\\-3\end{pmatrix}=-3\begin{pmatrix}-1\\1\\1\end{pmatrix}$. A normal to p_2 is $\begin{pmatrix}4\\1\\3\end{pmatrix}$. Since $\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\3\end{pmatrix}=-4+1+3=0$, p_1 is perpendicular to p_2.	

4(ii)	**Show $(1,0,-1)$ lies on *l*.** Equation of p_1 is $\underset{\sim}{r}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=-2$ Since $\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=-1+0-1=-2$, $(1,0,-1)$ lies on p_1. Since $4(1)+0+3(-1)=1$, $(1,0,-1)$ lies on p_2. Since $(1,0,-1)$ lies on both planes, it lies on *l*.	
	Find equation of line Line *l* is parallel to $\begin{pmatrix}-1\\1\\1\end{pmatrix}\times\begin{pmatrix}4\\1\\3\end{pmatrix}=\begin{pmatrix}2\\7\\-5\end{pmatrix}$. Equation of $l: \mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+t\begin{pmatrix}2\\7\\-5\end{pmatrix}, t\in\mathbb{R}$	
	<u>**Alternative #1**</u> $p_1:\mathbf{r}.\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\3\\-5\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix} \Rightarrow p_1:-x+y+z=-2 \quad ---(1)$ $p_2:4x+y+3z=1 \quad ---(2)$ Solving (1) and (2) simultaneously, by GC, equation of *l*: $\mathbf{r}=\begin{pmatrix}3/5\\-7/5\\0\end{pmatrix}+s\begin{pmatrix}-2/5\\-7/5\\1\end{pmatrix}, s\in\mathbb{R}$ $\mathbf{r}=\begin{pmatrix}3/5\\-7/5\\0\end{pmatrix}+t\begin{pmatrix}-2\\-7\\5\end{pmatrix}, t\in\mathbb{R}$	

(iii)	Let C be the point $(1,0,-1)$ on l and F be the foot of perpendicular of B to l. $\overrightarrow{BC}=\begin{pmatrix}1\\0\\-1\end{pmatrix}-\begin{pmatrix}2\\2\\-3\end{pmatrix}=\begin{pmatrix}-1\\-2\\2\end{pmatrix}$ $\overrightarrow{BF}=\left(\overrightarrow{BC}\cdot\frac{\mathbf{n}_1}{\lvert\mathbf{n}_1\rvert}\right)\frac{\mathbf{n}_1}{\lvert\mathbf{n}_1\rvert}$, where $\mathbf{n}_1$ is a normal to p_1 $=\frac{\left(\overrightarrow{BC}\cdot\mathbf{n}_1\right)\mathbf{n}_1}{\lvert\mathbf{n}_1\rvert^2}$ $=\frac{1}{3}\left(\begin{pmatrix}-1\\-2\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}\right)\begin{pmatrix}-1\\1\\1\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\begin{pmatrix}0\\3\\-5\end{pmatrix}-\begin{pmatrix}2\\2\\-3\end{pmatrix}=\begin{pmatrix}-2\\1\\-2\end{pmatrix}$ Area of $\triangle ABB'=2\times\left(\text{Area of }\triangle ABF\right)$ $=2\times\frac{1}{2}\left\lvert\overrightarrow{BA}\times\overrightarrow{BF}\right\rvert$ $=\left\lvert\begin{pmatrix}-2\\1\\-2\end{pmatrix}\times\frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rvert=\frac{1}{3}\left\lvert\begin{pmatrix}3\\4\\-1\end{pmatrix}\right\rvert$ $=\frac{\sqrt{26}}{3}$ units^2	

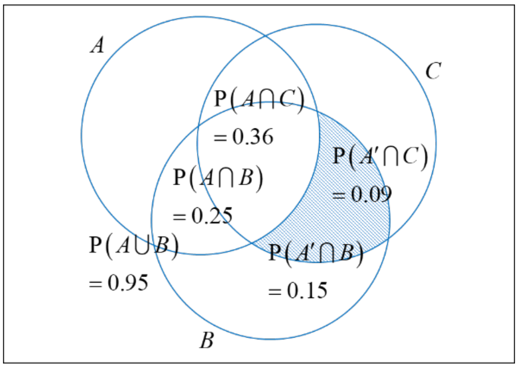
	Alternative (finding $\overrightarrow{BF}$ using $\overrightarrow{CF}$) $\overrightarrow{CB}=\begin{pmatrix}2\\2\\-3\end{pmatrix}-\begin{pmatrix}1\\0\\-1\end{pmatrix}=\begin{pmatrix}1\\2\\-2\end{pmatrix}$ $\overrightarrow{CF}=\dfrac{\left(\overrightarrow{CB}.\begin{pmatrix}2\\7\\-5\end{pmatrix}\right)\begin{pmatrix}2\\7\\-5\end{pmatrix}}{4+49+25}$ $=\dfrac{1}{78}\left(\begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}2\\7\\-5\end{pmatrix}\right)\begin{pmatrix}2\\7\\-5\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}2\\7\\-5\end{pmatrix}$ $\overrightarrow{BF}=\overrightarrow{CF}-\overrightarrow{CB}=\dfrac{1}{3}\begin{pmatrix}2\\7\\-5\end{pmatrix}-\begin{pmatrix}1\\2\\-2\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}$	
(iv)	Q lies p_1 and p_2, hence Q lies in the line l.	
(v)	$\left[\begin{pmatrix}1\\0\\-1\end{pmatrix}+t\begin{pmatrix}2\\7\\-5\end{pmatrix}\right].\begin{pmatrix}a\\3\\1\end{pmatrix}=-9$ $a-1+(2a+16)t=-9$ --------(*) <u>Case 1</u>: If $a=-8$, (*) is true for all values of t. <u>Case 2</u>: If $a\neq-8$, $t=-\dfrac{a+8}{2a+16}=-\dfrac{1}{2}$. $\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\left(-\dfrac{1}{2}\right)\begin{pmatrix}2\\7\\-5\end{pmatrix}=\begin{pmatrix}0\\-3.5\\1.5\end{pmatrix}$ In both cases, we have $(0,-3.5,1.5)$ lying on both l and p_3. Hence, coordinates of Q is $(0,-3.5,1.5)$.	
5i	Number of ways $=\dfrac{8!}{3!}$ $=6720$	
5ii	Different cases: SxxxxSxxxxxS, SxxxxxSxxxxS, xSxxxxSxxxxS, and SxxxxSxxxxSx. Number of ways $=4\times\dfrac{9!}{2!2!}$ $=362880$	

	$\mathrm{P}(\text{all 8 letters distinct})=\dfrac{{}^2C_1\times{}^3C_1\times{}^2C_1\times{}^5C_5}{{}^{12}C_8}$ $=\dfrac{4}{165}$ **<u>Alternative</u>** $\mathrm{P}(\text{all 8 letters distinct})$ $=\left(\frac{2}{12}\right)\left(\frac{3}{11}\right)\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)\left(\frac{1}{8}\right)\left(\frac{1}{7}\right)\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\times 8!$ $=\dfrac{4}{165}$	
6i	$\mathrm{P}(X=1)=\dfrac{1}{3}$ $\mathrm{P}(X=2)=\dfrac{2}{3}\times\dfrac{1}{2}=\dfrac{1}{3}$ $\mathrm{P}(X=3)=\dfrac{2}{3}\times\dfrac{1}{2}\times 1=\dfrac{1}{3}$ 	

6ii	(ii) $\mathrm{P}(X=x)=\frac{1}{N}$, for $x = 1, 2, \ldots, N$ $\mathrm{E}(X)=\frac{N+1}{2}$ (by symmetry) <u>OR</u> $\mathrm{E}(X)=1\cdot\frac{1}{N}+2\cdot\frac{1}{N}+\cdots+N\cdot\frac{1}{N}$ $=\frac{1}{N}(1+2+\cdots N)$ $=\frac{1}{N}\left(\frac{N}{2}(1+N)\right)$ $=\frac{N+1}{2}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=\sum_{x=1}^{N}x^2\mathrm{P}(X=x)-\left(\frac{N+1}{2}\right)^2$ $=\frac{1}{N}\sum_{x=1}^{N}x^2-\left(\frac{N+1}{2}\right)^2$ $=\frac{1}{N}\left[\frac{N(N+1)(2N+1)}{6}\right]-\left(\frac{N+1}{2}\right)^2$ $=\frac{N+1}{12}\left[2(2N+1)-3(N+1)\right]$ $=\frac{(N+1)(N-1)}{12}$	
7i	$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $0.95=0.8+0.4-\mathrm{P}(A\cap B)$ $\mathrm{P}(A\cap B)=0.25$ A and B are not independent because $\mathrm{P}(A\cap B)=0.25\neq 0.32=(0.8)(0.4)=\mathrm{P}(A)\mathrm{P}(B)$.	
7ii	$0.2<c\leq 1$  Think: If $\mathrm{P}(C)\leq 0.2$ then it is possible for $C\subseteq A'$. This means it is still possible for A and C to not intersect. But when $\mathrm{P}(C)>0.2$, this scenario won't happen!	

7iii

$$\begin{aligned} \mathrm{P}(A \cap C) &= \mathrm{P}(C \mid A)\mathrm{P}(A) \\ &= (0.45)(0.8) \\ &= 0.36 \end{aligned}$$



Note that $P(A \cap B) = 0.25,\ P(A \cap B') = 0.15$

Note that $P\left((A \cup B)'\right) = 1 - 0.95 = 0.05$ and

$$\begin{aligned} P(A' \cap C) &= P(C) - P(A \cap C) \\ &= 0.45 - 0.36 = 0.09 \end{aligned}.$$

Hence,

$\min\left(\mathrm{P}(A' \cap B \cap C)\right) = 0.09 - 0.05 = 0.04$ and

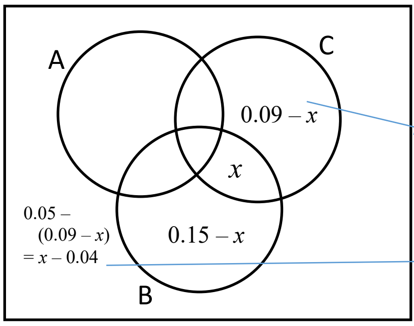
$\max\left(\mathrm{P}(A' \cap B \cap C)\right) = 0.09$ which occurs when

$(A' \cap C) \subseteq (A' \cap B)$.

Alternative:

Let $\mathrm{P}(A' \cap B \cap C) = x$. Note that

$P(A \cap B) = 0.25,\ P(A \cap B') = 0.15$ and $\mathrm{P}(A \cap C) = 0.36$. We also know that $P\left((A \cup B)'\right) = 1 - 0.95 = 0.05$. Using these to fill up the venn diagram below:



Since all probabilities are at least 0 and at most 1, we must have

max value of x = 0.09

and

min value of x = 0.04

8(a)

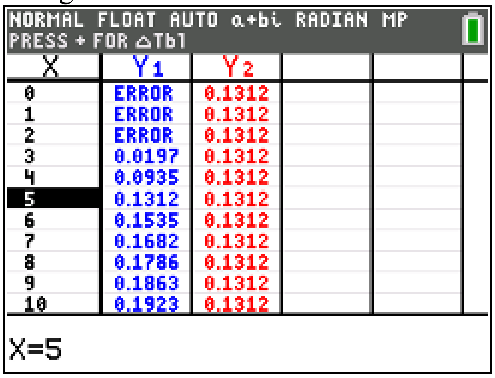
$\mathrm{B}(n, p)$

Given: $\mathrm{E}(X) = 2.75 \quad \Rightarrow np = 2.75 \quad ---(1)$

$\mathrm{P}(X < 2) = \mathrm{P}(X \le 1) = 0.13122$

<u>Method 1:</u>

Using GC:



Hence $n = 5$

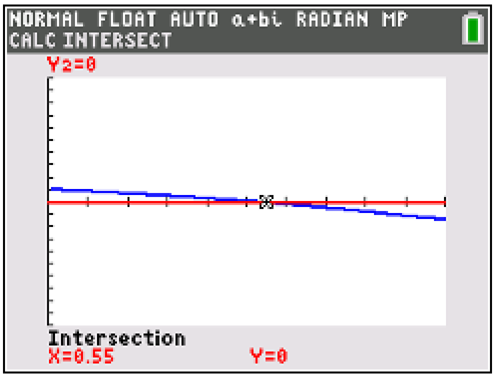
$$p = \frac{2.75}{5} = 0.55$$

<u>Method 2:</u>

$\mathrm{P}(X = 0) + \mathrm{P}(X = 1) = 0.13122$

$(1-p)^n + np(1-p)^{n-1} = 0.13122$

From (1): $(1-p)^{\frac{2.75}{p}} + 2.75(1-p)^{\frac{2.75}{p}-1} = 0.13122$



From the GC: $p = 0.55$

$$\therefore n = \frac{2.75}{0.55} = 5$$

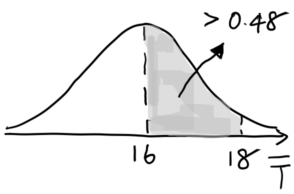
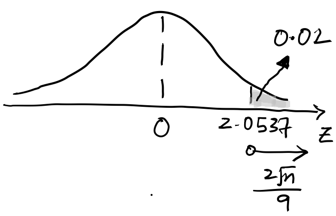
bi

(1) The probability that a resident uses Sharebike is $\frac{a}{100}$ for all the n residents.

(2) The residents use Sharebike independently.

bii	$X \sim \mathrm{B}(45, 0.08)$ $\mathrm{P}(9 \le X \le 13) = \mathrm{P}(X \le 13) - \mathrm{P}(X \le 8)$ $= 0.00843$	
biii	$X \sim \mathrm{B}(12, r)$, where $r = \dfrac{a}{100}$ $\mathrm{P}(X=1) < \mathrm{P}(X=2)$ and $\mathrm{P}(X=2) > \mathrm{P}(X=3)$ $\binom{12}{1} r(1-r)^{11} < \binom{12}{2} r^2(1-r)^{10}$ and $\binom{12}{2} r^2(1-r)^{10} > \binom{12}{3} r^3(1-r)^9$ $r(1-r)^{10}\left[12(1-r)-66r\right] < 0$ and $r^2(1-r)^9\left[66(1-r)-220r\right] > 0$ Since $(1-r) > 0$ and $r > 0$, $12 - 78r < 0$ and $66 - 286r > 0$ $r > \dfrac{2}{13}$ and $r < \dfrac{3}{13}$ $\therefore \dfrac{2}{13} < r < \dfrac{3}{13}$ $\therefore \dfrac{200}{13} < a < \dfrac{300}{13}$	
9 (i)	Let $w = x - 425$ $\bar{x} = \bar{w} + 425 = 425 - \dfrac{136}{40} = 421.6$ (exact value) $s_x^2 = s_w^2 = \dfrac{1}{39}\left(4927.5 - \dfrac{(-136)^2}{40}\right) = \dfrac{4465.1}{39} = 114.49 \approx 114$	
(ii)	Let μ g be the population mean. $\mathrm{H}_0 : \mu = 425$ $\mathrm{H}_1 : \mu \ne 425$ Level of Significance: 5% Test Statistic: Since n = 40 is large, by Central Limit Theorem, $\overline{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\overline{X} - 425}{S/\sqrt{n}} \sim \mathrm{N}(0,1)$ approximately. Computation: p-value = 0.044466 (or z-value = –2.0097) Conclusion: Since p-value = 0.0445 < 0.05, (or |–2.01| > 1.96), H_0 is rejected at 5% level of significance. Hence there is **sufficient evidence to conclude that the mean mass of a can of beans is not 425g.**	
(iii)	No assumption is needed. Since the sample size is large, by Central Limit Theorem, the distribution of the **sample mean mass** of a can of beans, $\overline{X}$, is **approximately** normal.	
(iv)	There is a probability of 0.05 that the test concludes that the mean mass of a can of beans is not 425g which it is actually 425g.	

(v)	Let Y g be the mass of a packet of frozen corn and μ_Y g be the population mean. $H_0 : \mu_Y = 380$ $H_1 : \mu_Y < 380$ Level of Significance: $\alpha\,\%$ Test Statistic: When H_0 is true, $Z = \dfrac{\overline{Y} - 380}{12/\sqrt{n}} \sim N(0,1)$ Computation: $\overline{y} = 375$, p-value = 0.053292 (or z-value = –1.6137) For H_0 not to be rejected, p-value > $\dfrac{\alpha}{100}$ $\Rightarrow 0.053292 > \dfrac{\alpha}{100}$ $\therefore \{\alpha \in \mathbb{R} : 0 < \alpha < 5.33\}$	
10 (i)	Let X and Y be the masses, in g, of a randomly chosen butter cookie and a randomly chosen chocolate cookie respectively. $X \sim N(15, 0.4^2)$ and $Y \sim N(20, 1.2^2)$ $P(X > 15.5) = 0.10565 \approx 0.106 \quad (\text{to 3 s.f.})$	
(ii)	Let W be the number of butter cookies (out of 10) with mass more than 15.5g. $W \sim N(10, 0.10565)$ $P(W \geq 4) = 1 - P(W \leq 3)$ $= 0.0155$	
(iii)	Let $S = \dfrac{6}{100}(X_1 + \ldots + X_{12}) + \dfrac{7.5}{100}(Y_1 + \ldots + Y_{12})$ $E(S) = \dfrac{6}{100}(12)(15) + \dfrac{7.5}{100}(12)(20)$ $= 28.8$ $\text{Var}(S) = \left(\dfrac{6}{100}\right)^2 (12)(0.4)^2 + \left(\dfrac{7.5}{100}\right)^2 (12)(1.2)^2$ $= 0.104112$ (exact value) $S \sim N(28.8, 0.104112)$ $P(S < 29) = 0.732$	

(iv)	We assume that the masses of cookies are independent of each other. Note: It is **insufficient** to say that “mass of a butter cookie is independent of the mass of a chocolate cookie”. This is because for the 12 butter cookies bought, we also need the condition that the mass of a butter cookie is independent of the mass of any other butter cookie. The same goes for the 12 chocolate cookies bought.	
(v)	Suppose T follows normal distribution i.e. $T \sim \mathrm{N}\left(16, 9^2\right)$ $\mathrm{P}\left(T < 0\right) = 0.037720$ which is not insignificant and time taken cannot be negative. Hence a normal distribution, with this mean and standard deviation, would not give a good approximation to the distribution of T. **Alternatively,** Due to the empirical rule of normal distribution, we would expect approximately 95% of the data to lie within 2 standard deviation from the mean. $16 \pm 2(9)$ have a range of values from -2 to 34. This would mean a non-negligible/significant 2.5% will have values below -2, which is impossible as time taken cannot be negative. Hence a normal distribution, with this mean and standard deviation, would not give a good approximation to the distribution of T.	
(vi)	Since $n \geq 30$, by Central Limit Theorem, $\overline{T} \sim \mathrm{N}\left(16, \frac{9^2}{n}\right)$ approximately. $\mathrm{P}\left(16 < \overline{T} < 18\right) > 0.48$ $\mathrm{P}\left(\overline{T} > 18\right) < 0.5 - 0.48 = 0.02$  Given $\mathrm{P}\left(Z > \frac{18-16}{9/\sqrt{n}}\right) < 0.02$ $\mathrm{P}\left(Z > \frac{2\sqrt{n}}{9}\right) < 0.02$ From the GC, $\mathrm{P}\left(Z > 2.0537\right) = 0.02$  $\frac{2\sqrt{n}}{9} > 2.0537$ $n > 85.4$ Least n = 86	

2021 JC2 H2 Mathematics Prelim P2 Solutions