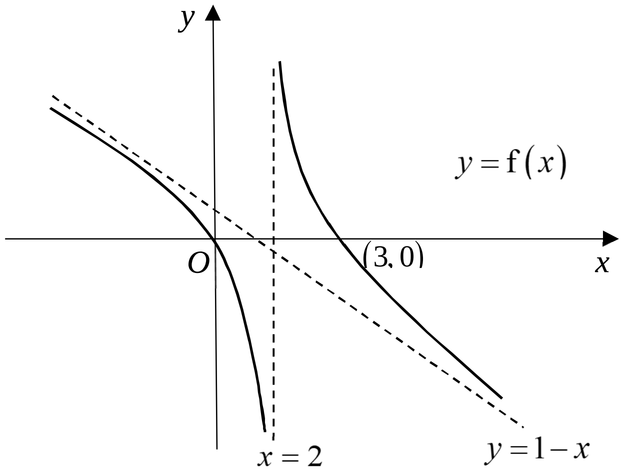
2021 HCI Prelim Paper 1

1 The diagram shows the graph of $y = \mathrm{f}(x)$. The lines $x = 2$ and $y = 1 - x$ are asymptotes to the curve, and the graph passes through the points $(0, 0)$ and $(3, 0)$.



Sketch the following graphs, indicating clearly the coordinates of any axial intercepts (where applicable) and the equations of any asymptotes.

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$ [3]

2 **(i)** The complex number z has modulus 2 and argument θ and the complex number w has modulus r and argument $-\dfrac{\pi}{2}$, where r is a positive constant and $-\dfrac{\pi}{2} < \theta < 0$. Given that $\dfrac{z}{w} = \dfrac{1}{32}\left(\sin\dfrac{\pi}{6} + \mathrm{i}\cos\dfrac{\pi}{6}\right)$, find the values of r and θ.

[3]

(ii) Hence find the 3 smallest positive integers n such that $\left(z^n\right)^*$ is purely imaginary. [3]

3 **(i)** Without using a calculator, solve the inequality $\dfrac{2x^2 + 3x}{2x^2 + x - 1} \le \dfrac{1}{2x + 2}$. [4]

(ii) Using your answer to part **(i)**, deduce the values of x for the inequality $\dfrac{2\cos^2 x + 3\cos x}{2\cos^2 x + \cos x - 1} \le \dfrac{1}{2\cos x + 2}$, where $-\pi \le x \le \pi$, leaving your answer in exact form. [3]

4 **(a)** Find $\displaystyle\int \frac{6\tan 3x}{1 + \cos 6x}\,\mathrm{d}x$. [3]

(b) It is given that $I_n = \displaystyle\int_1^{\mathrm{e}} (\ln x)^n \,\mathrm{d}x$, for $n \in \mathbb{Z}^+$.

(i) Use integration by parts to show that

$$I_n = \mathrm{e} - nI_{n-1}.$$ [2]

(ii) The region bounded by the curve $y=(\ln x)^2$, the x-axis and the line $x=\mathrm{e}$ is rotated through 2π radians about the x-axis. Using the result in **(b)(i)**, find the exact volume of the solid formed. [3]

Do not use a calculator for this question.

5 **(a)** The three complex numbers z_1, z_2 and z_3 are given as $z_1=2\mathrm{i}$, $z_2=2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ and $z_3=\frac{1}{7}\left(\cos\frac{2}{3}\pi+\mathrm{i}\sin\frac{2}{3}\pi\right)$. Find $\frac{1}{z_3}(z_2-z_1)$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

(b) The complex number z satisfies the equation $2z^3+az^2+18z-9=0$, where a is a real number. It is given that one root is of the form $z=k\mathrm{i}$, where k is real and negative. Find the values of a and k, and the other roots of the equation. [6]

6 A curve C is defined by the parametric equations

$$x=\frac{p}{t^2}-t,\quad y=pt+\tfrac{1}{t};$$

where t is a parameter, $t>0$, and p is a positive constant.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t(pt^2-1)}{-2p-t^3}$. [3]

The normal to C at $t=1$ passes through the point $A(3,-1)$.

(ii) Find the exact value of p. [3]

(iii) Given that the normal to C at $t=1$ meets the y axis at point B, find the length of AB. [3]

7 It is given that $\mathbf{a}$ and $\mathbf{b}$ are non zero vectors.

(a) Given that $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, by considering suitable scalar product, comment on the relationship between $\mathbf{a}$ and $\mathbf{b}$. [2]

(b) Given that $\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\mathbf{b}}{|\mathbf{b}|}$, what can you say about the relationship between $\mathbf{a}$ and $\mathbf{b}$? [1]

(c) Given that $2\mathbf{a}+\mathbf{b}=-5\mathbf{j}+\mathbf{k}$, $\mathbf{c}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$, $\mathbf{a.c}=5$ and $\mathbf{b}$ is a unit vector,

(i) find the value of $\mathbf{b.c}$. [2]

(ii) find the sine of the angle between $\mathbf{b}$ and $\mathbf{c}$. [2]

(iii) find $|\mathbf{b}\times\mathbf{c}|$, and state the geometrical meaning of this result. [2]

8 The function f is defined by $\mathrm{f}: x \mapsto \dfrac{2x+6}{x-2},\ x \in \mathbb{R}, x > 2$.

(i) Find f^{-1} in a similar form. [3]

(ii) Find the solution set for $\mathrm{f}^2(x) = x$. [1]

It is given that k is a constant such that $k > 3$.

The function g is defined by $\mathrm{g}: x \mapsto (x-5)^2 + k, x \in \mathbb{R}, x > 2$.

(iii) Show that fg exists. [1]

(iv) Find the range of values of fg, in terms of k. [2]

The function ϕ is defined by $\phi: x \mapsto \dfrac{2x+a}{x-2}$, where a is a constant, $x \in \mathbb{R}, x > 2$.

(v) Given that ϕ^{-1} exists, state the value that a cannot take, justifying your answer. [2]

A function h is said to be self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.

(vi) State the range of values of a such that ϕ is a self-inverse function. [1]

9 Given that $y = (\tan x + \sec x)^2$, show that $\cos x \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y$. [2]

(i) By repeat differentiation, find the Maclaurin series for y up to and including the term in x^3. [4]

(ii) Using the result obtained in (i) estimate the value of $\tan 1^{\circ} + \sec 1^{\circ}$ to 4 decimal places. [2]

(iii) By expressing y in terms of sine and cosine, use the standard series in MF26 to find the series expansion for y up to and including the term in x^3. [3]

(iv) Comment on the results obtained from part **(i)** and part **(iii)**. [1]

10 A flu virus is spreading in a community which has a fixed population of N people. Scientists discover that at time t weeks from the beginning, the rate of change of the number of people who are infected is proportional to the product of the number of people infected, I, and the number of people who are not infected.

(i) Write down a differential equation relating I and t. [1]

It is noted that at the beginning, 1% of the people in the community is infected with the virus. 12 weeks later, the proportion of people infected in the community has risen to 25%.

(ii) By solving the differential equation obtained in part **(i)**, show that

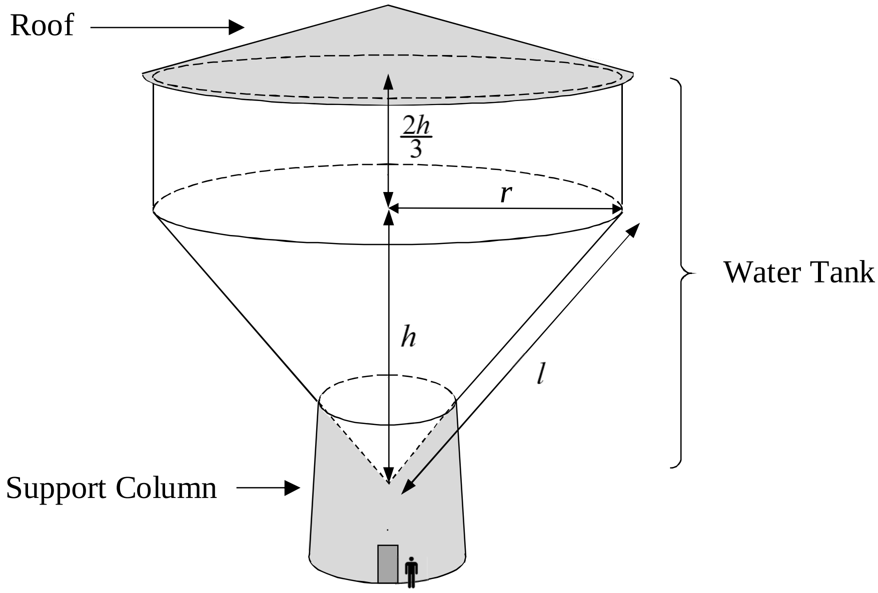
$$I = \frac{N}{99(33^{-\frac{t}{12}}) + 1}.$$ [6]

(iii) Find the least number of weeks needed for the proportion of people infected to exceed 50%, giving your answer to the nearest integer. [2]

(iv) Sketch the graph of I against t. Explain, with justification, what happens to the number of people infected if this situation continues indefinitely. [2]

(v) State a possible limitation of the model in part **(i)**. [1]

11 [It is given that a cone of radius r, slant height l and vertical height h has curved surface area πrl and volume $\frac{1}{3}\pi r^2 h$.]



A water tank is being constructed as a service reservoir supplying fully treated potable water to electronic chip manufacturing industries nearby. The water tank, held by a support column and covered by a roof, has a composite body made up of an inverted cone with radius r m, slant height l m and vertical height h m, and a cylinder of radius r m and height $\frac{2h}{3}$ m, as shown in the diagram.

It is assumed that the water tank is constructed with material of negligible thickness. It is given that the volume of the water tank is a fixed value v m^3 and the external surface area of the water tank, formed by the slanted surface of the inverted cone and the curved surface of the cylinder, is A m^2.

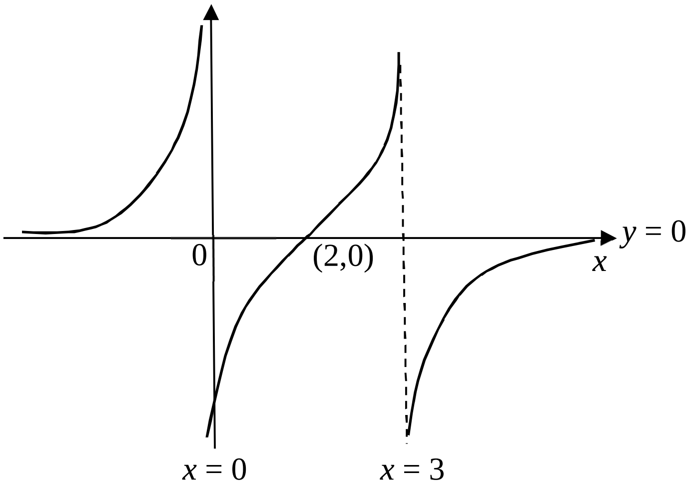
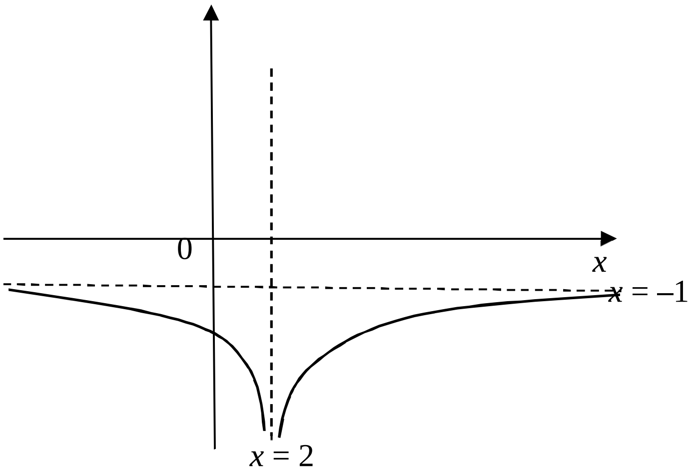
(i) Show that $A = \dfrac{4v}{3r} + \dfrac{1}{r}\sqrt{\pi^2 r^6 + v^2}$. [3]

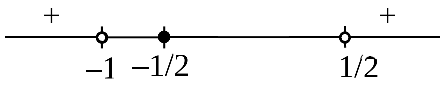
In order to keep the cost of construction down, A must be a minimum.

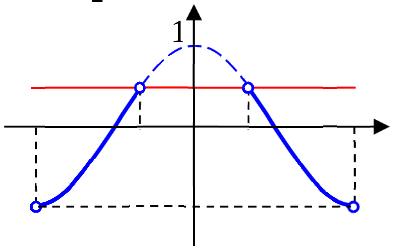
(ii) Use differentiation to show that when A is a minimum, $r^6 = k\left(\dfrac{v}{\pi}\right)^2$, where k is an exact constant to be determined. (You need not show that your answer gives a minimum.) [6]

(iii) Hence find the value of $\dfrac{h}{r}$ when A is a minimum. Comment on the significance of this value with reference to the shape of the water tank. [3]

2021 HCI Prelim Paper 1 Suggested Solutions

1(i)		
1(ii)		
2(i)	$z=2\mathrm{e}^{\mathrm{i}\theta}$, $w=r\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}$ $\frac{z}{w}=\frac{1}{32}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)=\frac{1}{32}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{3}\right)}$ $\left\lvert\frac{z}{w}\right\rvert=\frac{2}{r}=\frac{1}{32}$ $r=64$ $\arg\left(\frac{z}{w}\right)=\arg z-\arg w=\theta+\frac{\pi}{2}=\frac{\pi}{3}$ $\theta=\frac{\pi}{3}-\frac{\pi}{2}=-\frac{\pi}{6}$	

2(ii)	$z=2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$ $\left(z^n\right)^*=\left(2^n\mathrm{e}^{\mathrm{i}\left(-\frac{\pi n}{6}\right)}\right)^*$ $=2^n\mathrm{e}^{\mathrm{i}\left(\frac{\pi n}{6}\right)}$ $\frac{n\pi}{6}=k\pi+\frac{\pi}{2}$ $n=6k+3$ $n=3,9,15$	
3(i)	$\frac{2x^2+3x}{2x^2+x-1}\le\frac{1}{2x+2}$ $\frac{2x^2+3x}{(2x-1)(x+1)}-\frac{1}{2(x+1)}\le 0$ $\frac{4x^2+6x-(2x-1)}{2(2x-1)(x+1)}\le 0$ $\frac{4x^2+4x+1}{2(2x-1)(x+1)}\le 0$ $\frac{(2x+1)^2}{2(2x-1)(x+1)}\le 0$  $\therefore\ -1<x<\frac{1}{2}$	

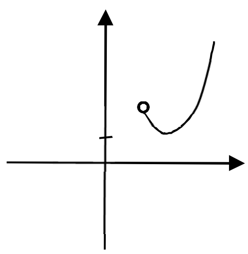
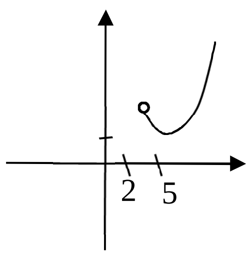
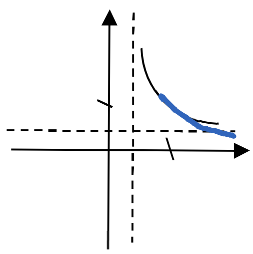
3(ii)	Replace x with $\cos x$, $\therefore\ -1<\cos x<\frac{1}{2}$  When $\cos x=\frac{1}{2}$, $\therefore\ x=\pm\frac{\pi}{3}$ Hence required values of x are $-\pi<x<-\frac{\pi}{3}$ or $\frac{\pi}{3}<x<\pi$	
4(a)	$\int\frac{6\tan 3x}{1+\cos 6x}\mathrm{d}x=\int\frac{6\tan 3x}{1+(2\cos^2 3x-1)}\mathrm{d}x$ $=\int\frac{6\tan 3x}{2\cos^2 3x}\mathrm{d}x$ $=\int 3\sec^2 3x\tan 3x\ \mathrm{d}x$ $=\frac{\tan^2 3x}{2}+C$ OR $\int\frac{3\sin 3x}{(\cos 3x)^3}\mathrm{d}x$ $=-\frac{(\cos 3x)^{-2}}{-2}+C$ $=\frac{1}{2}\sec^2 3x+C$	
4(b) (i)	$I_n=\int_1^{\mathrm{e}}(\ln x)^n\mathrm{d}x$ $=\left[x(\ln x)^n\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}xn(\ln x)^{n-1}\left(\frac{1}{x}\right)\mathrm{d}x$ $=[\mathrm{e}(\ln\mathrm{e})^n-0]-n\int_1^{\mathrm{e}}(\ln x)^{n-1}\mathrm{d}x$ $=\mathrm{e}-nI_{n-1}$ (Shown)	
4(ii)	Required volume $=\pi\int_1^{\mathrm{e}}(\ln x)^4\mathrm{d}x=\pi I_4$ <u>Method 1</u>: Using the result in (b)(i),	

	$I_4 = \mathrm{e} - 4I_3$ $I_3 = \mathrm{e} - 3I_2$ $I_2 = \mathrm{e} - 2I_1$ where $I_1 = \int_1^{\mathrm{e}} \ln x \, \mathrm{d}x$ $= \left[x \ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} x\left(\frac{1}{x}\right) \mathrm{d}x$ $= \mathrm{e} - \int_1^{\mathrm{e}} 1 \, \mathrm{d}x$ $= \mathrm{e} - \left[x\right]_1^{\mathrm{e}}$ $= \mathrm{e} - (\mathrm{e} - 1)$ $= 1$ Using the result in (b)(i), $\therefore I_4 = \mathrm{e} - 4I_3$ $= \mathrm{e} - 4(\mathrm{e} - 3I_2)$ $= \mathrm{e} - 4\mathrm{e} + 12(\mathrm{e} - 2I_1)$ $= 9\mathrm{e} - 24I_1$ $= 9\mathrm{e} - 24$ Thus, the required volume is $(9\mathrm{e} - 24)\pi$ units^3.	
5(a)	$z_2 - z_1 = 2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) - 2\mathrm{i}$ $= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) - 2\mathrm{i}$ $= \sqrt{3} + \mathrm{i} - 2\mathrm{i}$ $= \sqrt{3} - \mathrm{i}$ $\frac{z_2 - z_1}{z_3} = \frac{\sqrt{3} - \mathrm{i}}{\frac{1}{7}\left(\cos\frac{2}{3}\pi + \mathrm{i}\sin\frac{2}{3}\pi\right)}$ $= \frac{2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}}{\frac{1}{7}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}}$ $= 14\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3} - \frac{\pi}{6}\right)}$ $= 14\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$	

5(b)	$2z^3 + az^2 + 18z - 9 = 0$ Given that $z = k\mathrm{i}$ is a root $2(k\mathrm{i})^3 + a(k\mathrm{i})^2 + 18(k\mathrm{i}) - 9 = 0$ $-2k^3\mathrm{i} - ak^2 + 18k\mathrm{i} - 9 = 0$ $\left(-2k^3 + 18k\right)\mathrm{i} - 9 - ak^2 = 0$ ------------------- (1) From (1) $-2k^3 + 18k = 0$ $k(-2k^2 + 18) = 0$ $k = 0, k = -3$ or $k = 3$ Since k is real and negative, $k = -3$ $-9 - ak^2 = 0$ $-9 - 9a = 0$ $a = -1$ $2z^3 - z^2 + 18z - 9 = 0$ Since the coefficients are real, the complex numbers occur in conjugate pairs. One other root will be $z = 3\mathrm{i}$. $2z^3 - z^2 + 18z - 9 = (z - 3\mathrm{i})(z + 3\mathrm{i})(2z + b)$ By comparison $-9 = (-3\mathrm{i})(3\mathrm{i})(b)$ $b = -1$ The third root will be $z = \frac{1}{2}$.	

	The other roots of the equation $2z^3 - z^2 + 18z - 9$ are $z = 3\text{i}$, and $z = \frac{1}{2}$	
6(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{2p}{t^3} - 1$ $\frac{\mathrm{d}y}{\mathrm{d}t} = p - \frac{1}{t^2}$ $\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{pt^2-1}{t^2}}{\frac{-2p-t^3}{t^3}} = \frac{t(pt^2-1)}{-2p-t^3}$	
6(ii)	Gradient of normal $= -\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}} = \frac{2p+t^3}{t(pt^2-1)}$ $\therefore$ equation of normal at $t=1$ is $y-(p+1) = \frac{2p+1}{p-1}(x-(p-1))$ Since normal passes through $A(3,-1)$, $\therefore\quad -1-(p+1) = \frac{2p+1}{p-1}(3-(p-1))$ $(-2-p)(p-1) = (2p+1)(4-p)$ $-2p+2-p^2+p = 8p-2p^2+4-p$ $p^2-8p-2=0$ $\therefore\ p = \frac{8\pm\sqrt{64+8}}{2} = \frac{8\pm6\sqrt{2}}{2}$ $= 4+3\sqrt{2}$ (reject $4-3\sqrt{2}$ since $p>0$)	
6(iii)	When $t=1$ and $x=0$, equation of normal is $y-(p+1) = \frac{2p+1}{p-1}(0-(p-1))$ $y-p-1 = -2p-1$ $\therefore\ y = -p = -4-3\sqrt{2}$ Hence $B(0,-4-3\sqrt{2})$. $\therefore\ AB = \sqrt{(3-0)^2+(-1-(-4-3\sqrt{2}))^2}$ $= 7.839377789$ $= 7.84$ units (3 s.f.)	
7(a)	Note that for any vector $\underset{\sim}{v}$, $\underset{\sim}{v}\cdot\underset{\sim}{v} = \lvert\underset{\sim}{v}\rvert^2$. Hence we have $\lvert\underset{\sim}{a}+\underset{\sim}{b}\rvert^2 = \lvert\underset{\sim}{a}-\underset{\sim}{b}\rvert^2$ $\Rightarrow (\underset{\sim}{a}+\underset{\sim}{b})\cdot(\underset{\sim}{a}+\underset{\sim}{b}) = (\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b})$ $\Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{a}+2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{a}-2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}$ $\Rightarrow 2\underset{\sim}{a}\cdot\underset{\sim}{b}+2\underset{\sim}{a}\cdot\underset{\sim}{b} = \underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}-\underset{\sim}{a}\cdot\underset{\sim}{a}-\underset{\sim}{b}\cdot\underset{\sim}{b}$ $\Rightarrow 4\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ $\Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$	

	Hence $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$.	
7(b)	$\dfrac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}=\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $\Rightarrow \hat{\underset{\sim}{a}}=\hat{\underset{\sim}{b}}$ (the unit vector of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are the same) Hence $\underset{\sim}{a}$ and $\underset{\sim}{b}$ have the same direction (or they are parallel to each other).	
7(c)(i)	$(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{c}=\begin{pmatrix}0\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}$ $\Rightarrow 2\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c}=12$ $\Rightarrow 2(5)+\underset{\sim}{b}\cdot\underset{\sim}{c}=12$ $\Rightarrow \underset{\sim}{b}\cdot\underset{\sim}{c}=2$	
7(c)(ii)	$\cos\theta=\dfrac{\underset{\sim}{b}\cdot\underset{\sim}{c}}{\lvert\underset{\sim}{b}\rvert\lvert\underset{\sim}{c}\rvert}=\dfrac{2}{3},\quad \lvert\underset{\sim}{c}\rvert=\sqrt{1^2+2^2+2^2}=3,\quad \lvert\underset{\sim}{b}\rvert=1$ $\Rightarrow \cos\theta=\dfrac{2}{3}$ (hence θ is acute) $\therefore \sin\theta=\sqrt{1-\cos^2\theta}=\dfrac{\sqrt{5}}{3}$	
7(c)(iii)	$\lvert\underset{\sim}{b}\times\underset{\sim}{c}\rvert=\lvert\underset{\sim}{b}\rvert\lvert\underset{\sim}{c}\rvert\sin\theta=\sqrt{5}$ Let $\overrightarrow{OB}=\underset{\sim}{b}$ and $\overrightarrow{OC}=\underset{\sim}{c}$. And note that $\underset{\sim}{b}$ is unit vector. Geometrical meaning 1: $\lvert\underset{\sim}{b}\times\underset{\sim}{c}\rvert$ means the parallelogram with adjacent sides OB and OC has area $\sqrt{5}$ units square. Or Geometrical meaning 2: The perpendicular distance of the point $C(1,-2,2)$ to the line passing through origin O and with direction vector $\underset{\sim}{b}$ is $\sqrt{5}$ units.	

8(i)	$y=\dfrac{2x+6}{x-2}$ $xy-2y=2x+6$ $x(y-2)=2y+6$ $x=\dfrac{2y+6}{y-2}$ $\mathrm{f}^{-1}(x)=\dfrac{2x+6}{x-2}, x>2$ $\mathrm{f}^{-1}(x) \to \dfrac{2x+6}{x-2}, x>2$	
8(ii)	Notice from part (ii) that $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$, i.e. self-inverse. $\mathrm{f}^2(x)=x$ Apply inverse on both sides: $\Rightarrow \mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\Rightarrow$ Solution is when $\mathrm{D_f}=R_\mathrm{f}$ since it is self-inverse The solution is $\{x\in\mathbb{R}: x>2\}$.	
8(iii)	fg exists $\Leftrightarrow R_\mathrm{g}\subseteq D_\mathrm{f}$ $R_\mathrm{g}=[k,\infty)$ $D_\mathrm{f}=(2,\infty)$ Since $R_\mathrm{g}\subseteq D_\mathrm{f}$, as $k>3>2$, fg exists. 	
8(iv)	$\mathrm{g}(x)=(x-5)^2+k, x>2$   $(2,\infty)\xrightarrow{g}[k,\infty)\xrightarrow{f}(2,\dfrac{2k+6}{k-2}]$ $R_{fg}=(2,\dfrac{2k+6}{k-2}]$	

8(v)	$y=\dfrac{2x+a}{x-2}$ $2x+a\neq k\left(x-2\right)$ If $a=-4$, we have $\mathrm{f}\left(x\right)=2$, which is not a 1-1 function, thus ϕ^{-1} will not exist, a contradiction. Thus $a\neq-4$.	
8(vi)	$a>-4$	
9	$y=(\tan x+\sec x)^2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec^2 x+\sec x\tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(\tan x+\sec x)(\sec x+\tan x)$ $\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}=2y$	
9(i)	Diff wrt x: $-\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x}+\cos x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}y}{\mathrm{d}x}$ Diff wrt x: $-\cos x\dfrac{\mathrm{d}y}{\mathrm{d}x}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\cos x\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}-\sin x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ Sub $x=0$ $y=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=2,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=4,\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=10$ Let $y=\mathrm{f}(x)$ $y=\mathrm{f}(0)+\mathrm{f}\,'(0)x+\dfrac{\mathrm{f}\,''(0)}{2}x^2+\dfrac{\mathrm{f}\,'''(0)}{3!}x^3+\ldots$ $=1+2x+2x^2+\dfrac{5}{3}x^3+\ldots$	
9(ii)	$1^{\circ}=\dfrac{\pi}{180}$ Sub $x=\dfrac{\pi}{180}$ $\tan 1^{\circ}+\sec 1^{\circ}=\sqrt{1+2(\dfrac{\pi}{180})+2(\dfrac{\pi}{180})^2+\dfrac{5}{3}(\dfrac{\pi}{180})^3}$ $=\sqrt{1.03552468}$ $=1.0176$(4d.p.)	

9(iii)	$y=(\frac{\sin x+1}{\cos x})^2$ $=(1+\sin x)^2(\cos x)^{-2}$ $=(1+x-\frac{x^3}{6})^2(1-\frac{x^2}{2})^{-2}$ $=[1+2(x-\frac{x^3}{6})+(x-\frac{x^3}{6})^2](1+x^2+...)$ $=(1+2x+x^2-\frac{1}{3}x^3)(1+x^2+...)$ $=1+2x+2x^2+\frac{5}{3}x^3+...$	
9(iv)	Answers from both parts are consistent	
10(i)	$\frac{\mathrm{d}I}{\mathrm{d}t}=kI(N-I)$	
10(ii)	$\frac{\mathrm{d}t}{\mathrm{d}I}=\frac{1}{kI(N-I)}$ <u>Method 1</u>: Using partial fractions $\frac{\mathrm{d}t}{\mathrm{d}I}=\frac{1}{kI(N-I)}=\frac{1}{kN}\left(\frac{1}{I}+\frac{1}{N-I}\right)$ $t=\int\frac{1}{kN}\left(\frac{1}{I}+\frac{1}{N-I}\right)\mathrm{d}I$ $=\frac{1}{kN}\int\left(\frac{1}{I}+\frac{1}{N-I}\right)\mathrm{d}I$ $=\frac{1}{kN}\left[\ln I-\ln(N-I)\right]+C\ (\because 0<I<N)$ $=\frac{1}{kN}\ln\left(\frac{I}{N-I}\right)+C$ $kN(t-C)=\ln\left(\frac{I}{N-I}\right)$ $\mathrm{e}^{kNt-kNC}=\frac{I}{N-I}$ $A\mathrm{e}^{kNt}=\frac{I}{N-I}$ $A\mathrm{e}^{kNt}(N-I)=I$ $A\mathrm{e}^{kNt}N=I(1+A\mathrm{e}^{kNt})$ $I=\frac{A\mathrm{e}^{kNt}N}{1+A\mathrm{e}^{kNt}}=\frac{AN}{\mathrm{e}^{-kNt}+A}$	

	When $t=0,\ I=0.01N,\ 0.01N=\dfrac{AN}{1+A}$ $\Rightarrow 0.01(A+1)=A$ $\Rightarrow A=\dfrac{1}{99}$ When $t=12,\ I=0.25N,\ 0.25N=\dfrac{\frac{1}{99}N}{\mathrm{e}^{-12Nk}+\frac{1}{99}}$ $0.25\left(\mathrm{e}^{-12Nk}+\dfrac{1}{99}\right)=\dfrac{1}{99}$ $\Rightarrow \mathrm{e}^{-12Nk}=\dfrac{1}{33}$ $\Rightarrow Nk=\dfrac{\ln 33}{12}$ $\therefore I=\dfrac{\frac{1}{99}N}{\mathrm{e}^{\frac{-t\ln 33}{12}}+\frac{1}{99}}$ $=\dfrac{N}{99\mathrm{e}^{\frac{-t\ln 33}{12}}+1}$ $=\dfrac{N}{99\mathrm{e}^{\ln 33^{-\frac{t}{12}}}+1}$ $=\dfrac{N}{99\left(33^{-\frac{t}{12}}\right)+1}$ <u>Method 2</u>: Using MF26 $\dfrac{\mathrm{d}t}{\mathrm{d}I}=\dfrac{1}{-k\left(I^2-NI+\left(\frac{N}{2}\right)^2-\left(\frac{N}{2}\right)^2\right)}$ $=\dfrac{1}{k\left(\left(\frac{N}{2}\right)^2-\left(I-\frac{N}{2}\right)^2\right)}$ $t=\dfrac{1}{kN}\ln\left(\dfrac{\frac{N}{2}+I-\frac{N}{2}}{\frac{N}{2}-I+\frac{N}{2}}\right)+C$ $t=\dfrac{1}{kN}\ln\left(\dfrac{I}{N-I}\right)+C$	

	$kN(t-C)=\ln\left(\frac{I}{N-I}\right)$ $e^{kNt-kNC}=\frac{I}{N-I}$ $Ae^{kNt}=\frac{I}{N-I}$ $Ae^{kNt}(N-I)=I$ $Ae^{kNt}N=I(1+Ae^{kNt})$ $I=\frac{Ae^{kNt}N}{1+Ae^{kNt}}=\frac{AN}{e^{-kNt}+A}$ When $t=0,\ I=0.01N,\ 0.01N=\frac{AN}{1+A}$ $\Rightarrow 0.01(A+1)=A$ $\Rightarrow A=\frac{1}{99}$ When $t=12,\ I=0.25N,\ 0.25N=\frac{\frac{1}{99}N}{e^{-12Nk}+\frac{1}{99}}$ $0.25\left(e^{-12Nk}+\frac{1}{99}\right)=\frac{1}{99}$ $\Rightarrow e^{-12Nk}=\frac{1}{33}$ $\Rightarrow Nk=\frac{\ln 33}{12}$ $\therefore I=\frac{\frac{1}{99}N}{e^{\frac{-t\ln 33}{12}}+\frac{1}{99}}=\frac{N}{99e^{\frac{-t\ln 33}{12}}+1}=\frac{N}{99e^{\ln 33^{-\frac{t}{12}}}+1}$ $=\frac{N}{99\left(33^{-\frac{t}{12}}\right)+1}$	
10(iii)	<u>Method 1</u>:	

When $I > 0.50N$

$$\frac{N}{99\left(33^{-\frac{t}{12}}\right)+1} > 0.5N$$

$$99\left(33^{-\frac{t}{12}}\right)+1 < 2$$

$$99\left(33^{-\frac{t}{12}}\right) < 1$$

$$\left(33^{-\frac{t}{12}}\right) < \frac{1}{99}$$

$$\frac{-t\ln 33}{12} < -\ln 99$$

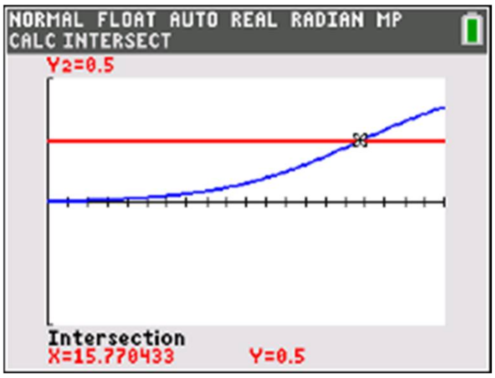
$t > 15.8$

The least number of weeks is 16.

<u>Method 2</u>: Use GC

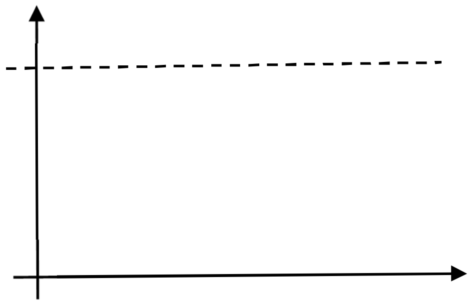
$$I = \frac{N}{99\left(33^{-\frac{t}{12}}\right)+1} > 0.5N$$

$$\frac{1}{99\left(33^{-\frac{t}{12}}\right)+1} - 0.5 > 0$$



From the graph, $t > 15.8$

The least number of weeks is 16.

10(iv)	 As $t \to \infty$, $\mathrm{e}^{-\frac{t\ln 33}{12}} \to 0 \Rightarrow I \to N$. If this situation continues indefinitely, almost everyone in the community will be infected by the virus.	
10(v)	One possible limitation is that the model assumes that everyone will get the virus which in actual fact, some may have natural immunity against the flu virus. OR: The death may have occurred during some weeks and N is no longer constant.	
11(i)	Volume of tank $v = \frac{1}{3}\pi r^2 h + \pi r^2 (\frac{2h}{3}) = \pi r^2 h$ $\therefore\ h = \frac{v}{\pi r^2}$ Let l m be slant height of cone. $\therefore\ l = \sqrt{r^2 + h^2}$ Let A m^2 be external surface area of water tank. $\therefore\ A = 2\pi r(\frac{2h}{3}) + \pi r l$ $= \frac{4}{3}\pi r h + \pi r\sqrt{r^2 + h^2}$ $= \frac{4}{3}\pi r(\frac{v}{\pi r^2}) + \pi r\sqrt{r^2 + (\frac{v}{\pi r^2})^2}$ $= \frac{4v}{3r} + \frac{1}{r}\sqrt{\pi^2 r^6 + v^2}$ (shown)	
11(ii)	$\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{4v}{3r^2} + \frac{r(\frac{1}{2})(\pi^2r^6+v^2)^{-\frac{1}{2}}(6\pi^2r^5)-(\pi^2r^6+v^2)^{\frac{1}{2}}}{r^2}$ $= \frac{(-4v)(\sqrt{\pi^2r^6+v^2})+9\pi^2r^6-3(\pi^2r^6+v^2)}{3r^2\sqrt{\pi^2r^6+v^2}}$ When $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$,	

	$\therefore (-4v)(\sqrt{\pi^2 r^6 + v^2}) + 6\pi^2 r^6 - 3v^2 = 0$ $\sqrt{\pi^2 r^6 + v^2} = \frac{6\pi^2 r^6 - 3v^2}{4v}$ $\pi^2 r^6 + v^2 = \frac{36\pi^4 r^{12} - 36\pi^2 v^2 r^6 + 9v^4}{16v^2}$ $16\pi^2 v^2 r^6 + 16v^4 = 36\pi^4 r^{12} - 36\pi^2 v^2 r^6 + 9v^4$ $36\pi^4 r^{12} - 52\pi^2 v^2 r^6 - 7v^4 = 0$ Or $\frac{\mathrm{d}A}{\mathrm{d}r} = -\frac{4v}{3r^2} + \frac{1}{2}\frac{1}{\sqrt{\pi^2 r^4 + \frac{v^2}{r^2}}}\left(4\pi^2 r^3 - \frac{2v^2}{r^3}\right)$ $= -\frac{4v}{3r^2} + \frac{2\pi^2 r^6 - v^2}{r^2\sqrt{\pi^2 r^6 + v^2}}$ $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$ $\frac{2\pi^2 r^6 - v^2}{r^2\sqrt{\pi^2 r^6 + v^2}} = \frac{4v}{3r^2}$ $6\pi^2 r^6 - 3v^2 = 4v\sqrt{\pi^2 r^6 + v^2}$ $36\pi^4 r^{12} - 52\pi^2 v^2 r^6 - 7v^4 = 0$ $\therefore \; r^6 = \frac{52\pi^2 v^2 \pm \sqrt{2704\pi^4 v^4 + 1008\pi^4 v^4}}{72\pi^4}$ $= \frac{52\pi^2 v^2 \pm \sqrt{3712}\pi^2 v^2}{72\pi^4}$ $= \frac{52\pi^2 v^2 \pm 8\sqrt{58}\pi^2 v^2}{72\pi^4}$ $= \left(\frac{13+2\sqrt{58}}{18}\right)\frac{v^2}{\pi^2}$ $\quad$ $[\left(\frac{13-2\sqrt{58}}{18}\right)\frac{v^2}{\pi^2}$ rejected since $r > 0]$	
11(iii)	Since $h = \frac{v}{\pi r^2}$, $\therefore \; \frac{h}{r} = \frac{v}{\pi r^3} = \frac{v}{\pi\left(\sqrt{\frac{13+2\sqrt{58}}{18}\frac{v}{\pi}}\right)}$ $= \frac{1}{\sqrt{\frac{13+2\sqrt{58}}{18}}}$ $= 0.7984889679$ $= 0.798 \quad (3 \text{ s.f.})$ For minimum A, the height of the inverted cone must be shorter than its radius, giving a water tank that is very wide at the top compared to its height.	

2021 HCI Prelim Paper 2
Section A: Pure Mathematics [40 marks]

1 A curve C has parametric equations

$$x = a\left(\frac{4}{\pi}t + \sin 2t\right) + 2a,$$
$$y = a\cos t,$$

where $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and a is a positive constant.

(i) Sketch C. State clearly the coordinates of any points where C meets the x-axis. [2]

(ii) Show that the area enclosed by C and the x-axis is given by

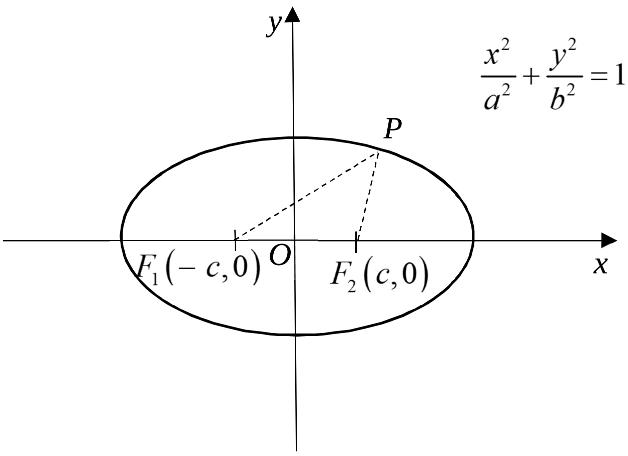
$$a^2\int_{\theta_1}^{\theta_2}\left(\frac{4}{\pi}\cos t + 2\cos t\cos 2t\right)\mathrm{d}t,$$

where θ_1 and θ_2 are values to be stated. [3]

(iii) Hence find, in terms of a, the exact area enclosed by C and the x-axis. [3]

2 **Clear workings and explanations are required for this question.**

(a) An ellipse of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $0 < b < a$, has two points called foci $F_1(-c, 0)$ and $F_2(c, 0)$. The definition of the ellipse is such that for every point P on the ellipse, the sum of the distance of P to F_1 and F_2 is always a constant k.



(i) By considering one of the x-intercepts of the ellipse, determine the value of k in terms of a and/or b. [2]

(ii) By considering another suitable point on the ellipse, find c in terms of a and b. [2]

(b) A hyperbola with equation $(y-h)^2-1=\frac{1}{4}(x-k)^2$ has $y=\frac{1}{2}x+\frac{3}{2}$ as one of its asymptotes, and the point $(1,3)$ is on the hyperbola. Find the values of h and k. [4]

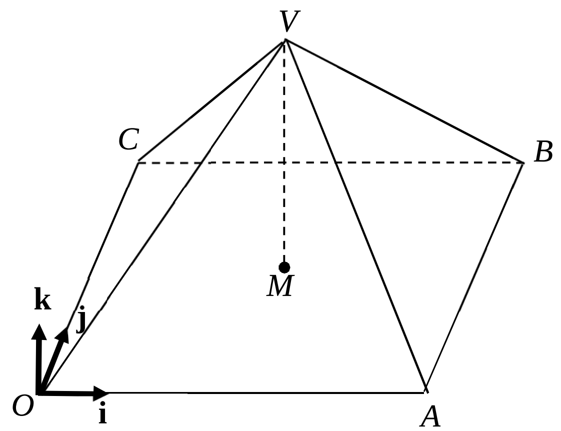
3 **(i)** Show that $\dfrac{4-x}{x(x-1)(x-2)}=\dfrac{2}{x}-\dfrac{3}{x-1}+\dfrac{1}{x-2}$. [1]

(ii) Hence find $\displaystyle\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}$ in terms of N, giving your answer in the form $\mathrm{f}(N)-k$, where k is a constant to be determined. [3]

(iii) Show that, for all integers $N\geq 4$, $\displaystyle\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}>-\frac{1}{6}$. [2]

(iv) Using your answer in part **(ii)**, find $\displaystyle\sum_{x=a}^{2a}\frac{3-x}{x(x+1)(x-1)}$, where a is an integer greater than 4, giving your answer in terms of a. (There is no need to express your answer as a single algebraic fraction.) [4]

4 Taking the point O as the origin, the diagram below shows a right pyramid with rectangular base $OABC$, and the base has its centre at M. The vertex of the pyramid is at V. The perpendicular unit vectors **i**, **j** and **k** are parallel to OA, OC and MV respectively. The length of OA, OC and MV are 8 units, 6 units and 10 units respectively.



(i) A line l with equations $\dfrac{y}{4}=\dfrac{z+1}{3}, x=1$, cuts the plane $OABC$ at the point P. Find the coordinates of P. [3]

(ii) Find a vector that is perpendicular to the plane OCV. [1]

(iii) Find the shortest distance from P to the plane OCV. Hence, or otherwise, find shortest the distance from P to the plane ABV. [4]

(iv) Write down the equation of the line where plane OCV meets plane ABV. [1]

(v) A point Q with coordinates (a,b,c) lies on the plane OCV, what can you say about the values of a, b and c? [2]

(vi) A point R with coordinates $(h,k,3)$ lies on the line segment CV, find the values of h and k. [3]

Section B: Probability and Statistics [60 marks]

5 Correlation and Regression [7]

6 Tom has a bag of wooden rectangular blocks of identical size. The bag contains 1 blue block, m red blocks and $(m-1)$ yellow blocks, where $m > 2$. Tom and Jerry play a game using Tom's bag of wooden blocks. Jerry draws 2 blocks at random, one at a time, without replacement. 3 points will be awarded if a yellow block is drawn, 2 points will be awarded if a red block is drawn, but no points will be awarded if a blue block is drawn. Jerry's final score is the product of the points awarded for the 2 blocks drawn.
Let the random variable X denotes Jerry's final score.

(i) Show that $\mathrm{P}(X=0)=\dfrac{1}{m}$ and hence, find the probability distribution of X. [4]

(ii) Find the value of m if Jerry's expected final score is 5. [2]

Tom pays Jerry \$5 if Jerry's final score is at least 5 and Jerry pays Tom \$$a$ if his final score is less than 5.

(iii) Using the value of m found in **(ii)**, find the range of values of a if Tom is expected to make a profit. [2]

7 A toy factory manufactures gel beads which are polymer beads that increase in size when soaked in water. On average, 8% of the gel beads are defective. The gel beads are packed in bags of 500. A significant number of customers recently gave feedback that many of the gel beads they bought could not expand in water or cracked while expanding. The quality control department decides to take a random sample of 20 gel beads from each bag to test. If more than 4 gel beads are found to be defective in the sample of 20, the bag is rejected. Otherwise the bag is accepted.

(i) State, in context, two assumptions needed for the number of defective gel beads in the sample to be well modelled by a binomial distribution. [2]

Assume now that the number of defective gel beads in a sample of 20 is modelled by a binomial distribution.

(ii) Find the probability that a randomly chosen bag of gel beads is rejected. [1]

(iii) An officer from the quality control department is in charge of inspecting 10 randomly chosen bags of gel beads. Find the probability that the last bag inspected is the second bag that is being rejected. [2]

(iv) A random sample of 20 gel beads is taken from a particular bag. Given that the bag is rejected, find the probability that there are more than 13 gel beads with no defects in the random sample of 20 gel beads. [3]

(v) The quality control department now decides to test 50 randomly chosen bags of gel beads. Find the probability that the mean number of defective gel beads found in the sample of each bag will not exceed 1.5. [2]

8 Mr Wong works as the Information Technology manager at a company. To boost the security of the network used by the company, he bought a breach detection system (BDS) which is a defensive tool designed to detect the activity of malware inside a network. The BDS sends out an alert to indicate that a breach has occurred. If there is malicious activity, there is a 90% chance that the BDS will correctly identify the activity as malicious. If the activity is not malicious, there is a 1% chance that the BDS will incorrectly identify the activity as malicious.

It is known that the BDS sends out alerts identifying 109 activities as malicious out of 10000 activities for a particular network system at a particular instance.

(i) Find the probability of an activity being malicious. [2]

(ii) Find the probability that the BDS is correct in detecting the type of activity in the network. Give your answer correct to 4 significant figures. [2]

(iii) Given that the BDS sends out an alert, find the probability that the activity is not malicious. Give your answer correct to 4 significant figures. [2]

(iv) Comment on the effectiveness of the BDS in the identification of a security breach. [1]

(v) The BDS is able to provide daily reports on the activities on the network which can only be accessible to Mr Wong. During the initial set-up, Mr Wong is required to set a 4-letter code formed from the letters of the word 'NINETEEN' as his password for authentication. Find the number of possible 4-letter codes that Mr Wong can choose from to be his password. [4]

9 **In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

Mr Tan manages a stall on his own, selling original egglets. Due to limited budget, he can only afford one egglet machine. As such, he can only prepare egglets one at a time. The time needed, in minutes, to make an original egglet follows the distribution $\mathrm{N}(4, 0.25)$.

(i) Sketch the distribution for the time needed to make an original egglet to be between 2 and 6 minutes. [2]

(ii) On a particular morning, Mr Tan sold 15 original egglets. Find the probability that there were exactly 6 original egglets which he took less than 4 minutes to make each of them. [2]

(iii) Find the probability that the time needed to make 3 randomly chosen original egglets differs from thrice the time needed to make a randomly chosen original egglet by not more than 5 minutes. [3]

Hoping to attract more customers, Mr Tan decides to introduce chicken floss egglets and soft drinks at his stall. The time needed, in minutes, to make a chicken floss egglet follows the distribution $\mathrm{N}(6, 1.2)$. The time needed, in minutes, to prepare a cup of soft drink is modelled as 20 % of the time needed to make an original egglet.

(iv) A customer orders an original egglet, a chicken floss egglet and a cup of soft drink. Given that the probability that the total time taken by Mr Tan to prepare these 3 items, one after another, in less than *m* minutes is at most 90%, find the range of possible values of *m*. [4]

(v) State an assumption needed for your calculations in part **(iv)** to be valid. [1]

10 Disposable face masks undergo the [1]Bacterial Filtration Efficiency (BFE) Test to assess how well a mask filters droplets containing biological agents such as bacteria or viruses. It is known that the higher the BFE, the more effective a mask is in preventing bacteria-containing droplets from reaching the wearer. For instance, a mask with a BFE of 95% will meet the requirements for medical and surgical masks since it blocks 95% of droplets it is exposed to.

A company that manufactures disposable face masks claimed that their masks are rated with BFE of at least 95%. An intern from the company wishes to check if the claim made by the company is valid. He is told that the BFE of the disposable face masks manufactured is distributed normally and that the standard deviation is 0.99%. He decides to carry out a hypothesis test at 5% level of significance on a random sample of 10 disposable face masks.

(i) Explain what is meant by a random sample in this context. [1]

(ii) State the hypotheses for the test, defining any symbols that you used, and find the set of possible mean BFE of the 10 randomly chosen disposable masks corresponding to the critical region. [4]

(iii) Given that the null hypothesis is rejected in the test conducted by the intern at 5% level of significance, comment if the same conclusion is obtained when the test is conducted at 1% level of significance. [1]

The company modified the manufacturing process to increase the BFE of the masks produced to be more than 98%. A quality control manager decides to perform a hypothesis test on a random sample of 70 disposable face masks produced by the modified manufacturing process to find out if this is the case.

The distribution of the BFE, y %, of the random sample of 70 disposable face masks produced by the modified manufacturing process are given as follows.

BFE, y %	95.8	96.8	97.5	98.3	98.5	98.7	98.8	99.2	99.5	99.8
Number of disposable face masks	3	8	11	10	10	9	5	5	7	2

(iv) Explain why the quality control manager takes a sample of 70 new masks but the intern only takes a sample of 10 masks. [2]

(v) Carry out a hypothesis test at 5% level of significance for the quality control manager. Give your conclusion in context. [4]

[1] A-STAR Explainer: Testing the Efficacy of Protective Face Masks.*https://www.a-star.edu.sg/News-and-Events/a-star-news/news/covid-19/*

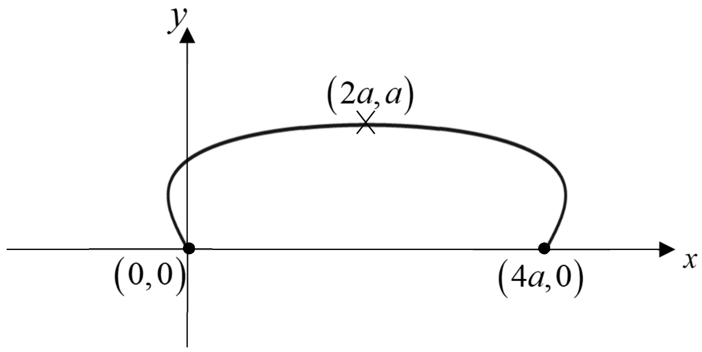
2021 HCI Prelim Paper 2 Suggested Solutions
Section A: Pure Mathematics [40 marks]

1 A curve C has parametric equations

$$x = a\left(\frac{4}{\pi}t + \sin 2t\right) + 2a,$$
$$y = a\cos t,$$

where $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ and a is a positive constant.

(i) Sketch C. State clearly the coordinates of any points where C meets the x-axis. [2]

Suggested Solutions	
 Note that the curve is symmetric when $t=0$ When $t=0$, $x=2a$ and $y=a\cos 0=a$ When $y=0$, $\cos t=0 \Rightarrow t=\pm\frac{\pi}{2}$. When $t=-\frac{\pi}{2}$, $x=a\left[\frac{4}{\pi}\left(-\frac{\pi}{2}\right)+\sin(-\pi)\right]+2a=0$ When $t=\frac{\pi}{2}$, $x=a\left(\frac{4}{\pi}\left(\frac{\pi}{2}\right)+\sin\pi\right)+2a=4a$	

(ii) Show that the area enclosed by C and the x-axis is given by

$$a^2\int_{\theta_1}^{\theta_2}\left(\frac{4}{\pi}\cos t + 2\cos t\cos 2t\right)\mathrm{d}t,$$

where θ_1 and θ_2 are values to be stated. [3]

Suggested Solutions	
Area enclosed by the curve and the x-axis $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} y\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t$ $=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a\cos t)\times a\left(\frac{4}{\pi}+2\cos 2t\right)\mathrm{d}t$	

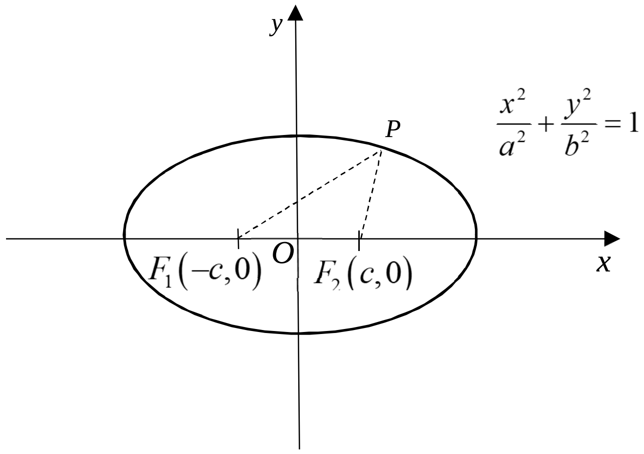
$= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t) \times \left(\frac{4}{\pi} + 2\cos 2t \right) \mathrm{d}t$ $= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{4}{\pi}\cos t + 2\cos t \cos 2t \right) \mathrm{d}t$ (Shown) where $\theta_1 = -\frac{\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$	

(iii) Hence find, in terms of a, the exact area enclosed by C and the axes. [3]

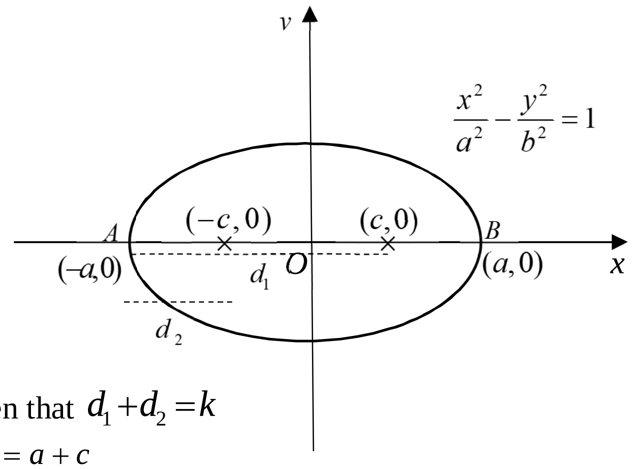
Suggested Solutions	
Area enclosed by the curve and the x-axis $= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{4}{\pi}\cos t + 2\cos t \cos 2t \right) \mathrm{d}t$ $= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{4}{\pi}\cos t + \cos 3t + \cos t \right) \mathrm{d}t$ $= a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(\frac{4}{\pi} + 1 \right)\cos t + \cos 3t \right) \mathrm{d}t$ $= a^2 \left[\left(\frac{4}{\pi} + 1 \right) \sin t + \frac{\sin 3t}{3} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ $= a^2 \left[\left(\frac{4}{\pi} + 1 - \frac{1}{3} \right) - \left(-\frac{4}{\pi} - 1 + \frac{1}{3} \right) \right]$ $= \left(\frac{8}{\pi} + \frac{4}{3} \right) a^2$ units^2	

2 Clear workings and explanations are required for this question.

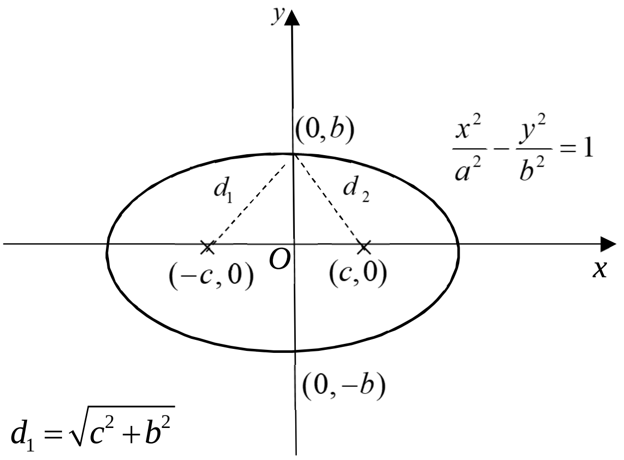
(a) An ellipse of equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $0<b<a$, has two points called foci $F_1(-c,0)$ and $F_2(c,0)$. The definition of the ellipse is such that for every point P on the ellipse, the sum of the distance of P to F_1 and F_2 is always a constant k.



(i) By considering one of the x-intercepts of the ellipse, determine the value of k in terms of a and/or b. [2]

Solution	
 Given that $d_1+d_2=k$ $d_1=a+c$ $d_2=a-c$ $d_1+d_2=2a$ $\therefore k=2a$	

(ii) By considering another suitable point on the ellipse, find c in terms of a and b. [2]

Solution	
 $d_1 = \sqrt{c^2 + b^2}$ $d_2 = \sqrt{c^2 + b^2}$ From (i) $d_1 + d_2 = 2a$, $d_1 = d_2 \Rightarrow d_1 = d_2 = a$ $\therefore \sqrt{c^2 + b^2} = a$ $c^2 + b^2 = a^2$ $c = \sqrt{a^2 - b^2}$	

(b) A hyperbola with equation $(y-h)^2 - 1 = \frac{1}{4}(x-k)^2$ has $y = \frac{1}{2}x + \frac{3}{2}$ as one of its asymptotes, and the point $(1,3)$ is on the hyperbola. Find the values of h and k. [4]

Solution	
Method 1 $h = \frac{1}{2}k + \frac{3}{2}$ ----- (1) $(3-h)^2 - 1 = \frac{1}{4}(1-k)^2$ ----- (2) $\left(3 - \frac{1}{2}k - \frac{3}{2}\right)^2 - 1 = \frac{1}{4}(1-k)^2$ $\left(\frac{3}{2} - \frac{1}{2}k\right)^2 - 1 = \frac{1}{4}(1-k)^2$ $\frac{1}{4}(3-k)^2 - \frac{4}{4} = \frac{1}{4}(1-k)^2$ $(3-k)^2 - 4 = (1-k)^2$ $9 - 6k + k^2 - 4 = 1 - 2k + k^2$ $\therefore k = 1$ $h = \frac{1}{2} + \frac{3}{2} = 2$ Method 2	

$(y-h)^2=1+\frac{(x-k)^2}{4}$ $y=h\pm\sqrt{1+\frac{(x-k)^2}{4}}$ When $x\to\pm\infty$ $y=h\pm\frac{x-k}{2}$ $y=h+\frac{x-k}{2}$ or $y=h-\frac{x-k}{2}$ $y=\frac{1}{2}x+h-\frac{k}{2}$ or $y=-\frac{1}{2}x+h+\frac{k}{2}$ $h-\frac{k}{2}=\frac{3}{2}$ $k=2h-3$--------- (1) Sub (1) into $(3-h)^2-1=\frac{1}{4}\left[1-(2h-3)\right]^2$ $8-6h+h^2=\frac{1}{4}(-2h+4)^2$ $8-6h+h^2=(h-2)^2$ $8-6h+h^2=h^2-4h+4$ $4=2h$ $h=2$ $k=4-3=1$	

3 **(i)** Show that $\dfrac{4-x}{x(x-1)(x-2)}=\dfrac{2}{x}-\dfrac{3}{x-1}+\dfrac{1}{x-2}$. [1]

Solution	
$\dfrac{2}{x}-\dfrac{3}{x-1}+\dfrac{1}{x-2}$ $=\dfrac{2\left(x^2-3x+2\right)-3\left(x^2-2x\right)+\left(x^2-x\right)}{x(x-1)(x-2)}$ $=\dfrac{4-x}{x(x-1)(x-2)}$	

(ii) Hence find $\displaystyle\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}$ in terms of N, giving your answer in the form $\mathrm{f}(N)-k$, where k is a constant to be determined. [3]

Solution	
$\displaystyle\sum_{x=4}^{N}\frac{4-x}{x(x-1)(x-2)}$ $\displaystyle=\sum_{x=4}^{N}\frac{2}{x}-\frac{3}{x-1}+\frac{1}{x-2}$ $=\left\{\begin{array}{lll} \frac{2}{4} & -\frac{3}{3} & +\frac{1}{2} \\ \frac{2}{5} & -\frac{3}{4} & +\frac{1}{3} \\ \frac{2}{6} & -\frac{3}{5} & +\frac{1}{4} \\ \frac{2}{7} & -\frac{3}{6} & +\frac{1}{5} \\ & \vdots & \\ \frac{2}{N-2} & -\frac{3}{N-3} & +\frac{1}{N-4} \\ \frac{2}{N-1} & -\frac{3}{N-2} & +\frac{1}{N-3} \\ \frac{2}{N} & -\frac{3}{N-1} & +\frac{1}{N-2} \end{array}\right.$ $=\dfrac{2}{N}-\dfrac{1}{N-1}-\dfrac{1}{6}$ $=\dfrac{N-2}{N(N-1)}-\dfrac{1}{6}$	

(iii) Show that, for all integers $N \geq 4$, $\sum_{x=4}^{N} \frac{4-x}{x(x-1)(x-2)} > -\frac{1}{6}$. [2]

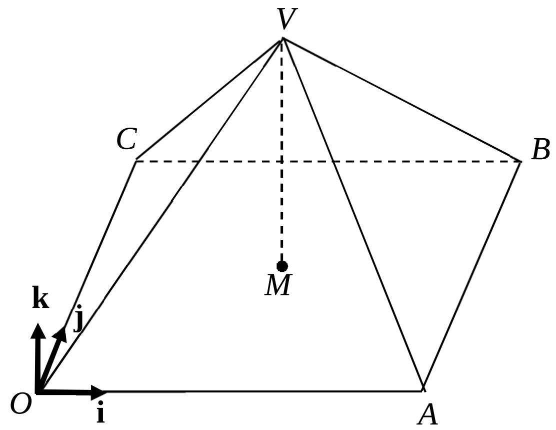
Solution	
$\sum_{x=4}^{N} \frac{4-x}{x(x-1)(x-2)} = \frac{2}{N} - \frac{1}{N-1} - \frac{1}{6}$ $= \frac{N-2}{N(N-1)} - \frac{1}{6}$ Since $N \geq 4$, then $\frac{N-2}{N(N-1)} > 0.$ $\sum_{x=4}^{N} \frac{4-x}{x(x-1)(x-2)} > -\frac{1}{6}$ (Shown)	

(iv) Using your answer in part **(ii)**, find $\sum_{x=a}^{2a} \frac{3-x}{x(x+1)(x-1)}$, where a is an integer greater than 4, giving your answer in terms of a. (There is no need to express your answer as a single algebraic fraction.) [4]

Solution	
$\sum_{x=a}^{2a} \frac{3-x}{x(x+1)(x-1)}$ Replace $x \to x-1$: $\sum_{x=a}^{2a} \frac{3-x}{x(x+1)(x-1)}$ $= \sum_{x-1=a}^{x-1=2a} \frac{3-(x-1)}{(x-1)(x)(x-2)}$ $= \sum_{x=a+1}^{2a+1} \frac{4-x}{x(x-1)(x-2)}$ $= \sum_{x=4}^{2a+1} \frac{4-x}{x(x-1)(x-2)} - \sum_{x=4}^{a} \frac{4-x}{x(x-1)(x-2)} \quad ---(*)$ $= \left[\frac{2}{2a+1} - \frac{1}{2a+1-1} - \frac{1}{6}\right] - \left[\frac{2}{a} - \frac{1}{a-1} - \frac{1}{6}\right]$ $= \frac{2}{2a+1} - \frac{5}{2a} + \frac{1}{a-1}$ Or from (*)	

$=\left[\dfrac{2a+1-2}{(2a+1)(2a)}-\dfrac{1}{6}\right]-\left[\dfrac{a-2}{a(a-1)}-\dfrac{1}{6}\right]$ $=\dfrac{2a-1}{(2a+1)(2a)}-\dfrac{a-2}{a(a-1)}$	

4 Taking the point O as the origin, the diagram below shows a right pyramid with rectangular base $OABC$, and the base has its centre at M. The vertex of the pyramid is at V. The perpendicular unit vectors **i**, **j** and **k** are parallel to OA, OC and MV respectively. The length of OA, OC and MV are 8 units, 6 units and 10 units respectively.



(i) A line l with equations $\dfrac{y}{4}=\dfrac{z+1}{3}, x=1$, cuts the plane $OABC$ at the point P. Find the coordinates of P. [3]

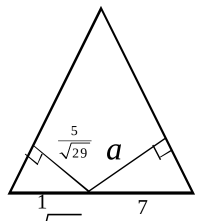
Solution	
$l:\dfrac{y}{4}=\dfrac{z+1}{3}, x=1$ $\Rightarrow l:\underset{\sim}{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}0\\4\\3\end{pmatrix}, \lambda\in\mathbb{R}$ Point P: $z=0\Rightarrow -1+3\lambda=0$ $\Rightarrow \lambda=\dfrac{1}{3}$ Coordinates of P is $\left(1,\dfrac{4}{3},0\right)$.	

(ii) Find a vector that is perpendicular to the plane OCV. [1]

Solution	

Normal to plane *OCV*: $\overrightarrow{OV}\times\overrightarrow{OC}=\begin{pmatrix}4\\3\\10\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=2\begin{pmatrix}-5\\0\\2\end{pmatrix}$	

(iii) Find the shortest distance from P to the plane *OCV*. Hence, or otherwise, find shortest the distance from P to the plane *ABV*. [4]

Solution	
Distance from P to *OCV* $=\dfrac{\left\lvert\overrightarrow{OP}.\begin{pmatrix}-5\\0\\2\end{pmatrix}\right\rvert}{\sqrt{29}}=\dfrac{\left\lvert\begin{pmatrix}1\\\frac{4}{3}\\0\end{pmatrix}.\begin{pmatrix}-5\\0\\2\end{pmatrix}\right\rvert}{\sqrt{29}}=\dfrac{5}{\sqrt{29}}$ or $\dfrac{5\sqrt{29}}{29}$ units Method 1: Consider a vertical plane containing P and parallel to $\mathbf{i}$. Using similar triangles: $\dfrac{a}{\frac{5}{\sqrt{29}}}=\dfrac{7}{1}$ $\Rightarrow a=\dfrac{35}{\sqrt{29}}$  Distance from P to *ABV*= $\dfrac{35}{\sqrt{29}}$ or $\dfrac{35\sqrt{29}}{29}$ units Method 2: Normal to plane *ABV*: $\overrightarrow{AV}\times\overrightarrow{AB}=\begin{pmatrix}-4\\3\\10\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-10\\0\\-4\end{pmatrix}$ Distance from P to *ABV* $=\dfrac{\left\lvert\overrightarrow{AP}.\begin{pmatrix}-10\\0\\-4\end{pmatrix}\right\rvert}{\sqrt{116}}=\dfrac{\left\lvert\begin{pmatrix}-7\\\frac{4}{3}\\0\end{pmatrix}.\begin{pmatrix}-10\\0\\-4\end{pmatrix}\right\rvert}{\sqrt{116}}=\dfrac{70}{\sqrt{116}}=\dfrac{35}{\sqrt{29}}$ or $\dfrac{35\sqrt{29}}{29}$ units	

(iv) Write down the equation of the line where plane *OCV* meets plane *ABV*. [1]

Solution	
Note that the line passes through V and parallel to $\mathbf{j}$	

$\underset{\sim}{r} = \begin{pmatrix} 4 \\ 3 \\ 10 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$	

(v) A point Q with coordinates (a, b, c) lies on the plane OCV, what can you say about the values of a, b and c? [2]

Solution	
$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \underset{\sim}{n} = 0$ $\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 0 \\ 2 \end{pmatrix} = 0$ $\Rightarrow -5a + 2c = 0, b \in \mathbb{R}$	

(vi) A point R with coordinates $(h, k, 3)$ lies on the line segment CV, find the values of h and k. [3]

Solution	
Method 1: C, R, V collinear, $\overrightarrow{CR} = \begin{pmatrix} h \\ k \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} h \\ k-6 \\ 3 \end{pmatrix}$ $\overrightarrow{CV} = \begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ $\overrightarrow{CR} = \alpha \overrightarrow{CV}$ $\Rightarrow h = 4\alpha$ $k - 6 = -3\alpha$ $3 = 10\alpha$ $\Rightarrow \alpha = \frac{3}{10}, h = \frac{6}{5}, k = \frac{51}{10}$ Method 2: Equation of line CV:	

$\underset{\sim}{r}=\begin{pmatrix}0\\6\\0\end{pmatrix}+\gamma\begin{pmatrix}4\\-3\\10\end{pmatrix},\gamma\in\mathbb{R}$ $\Rightarrow\begin{pmatrix}h\\k\\3\end{pmatrix}=\begin{pmatrix}4\gamma\\6-3\gamma\\10\gamma\end{pmatrix}$ $\Rightarrow\gamma=\frac{3}{10},h=\frac{12}{10}=\frac{6}{5},k=6-\frac{9}{10}=\frac{51}{10}$	

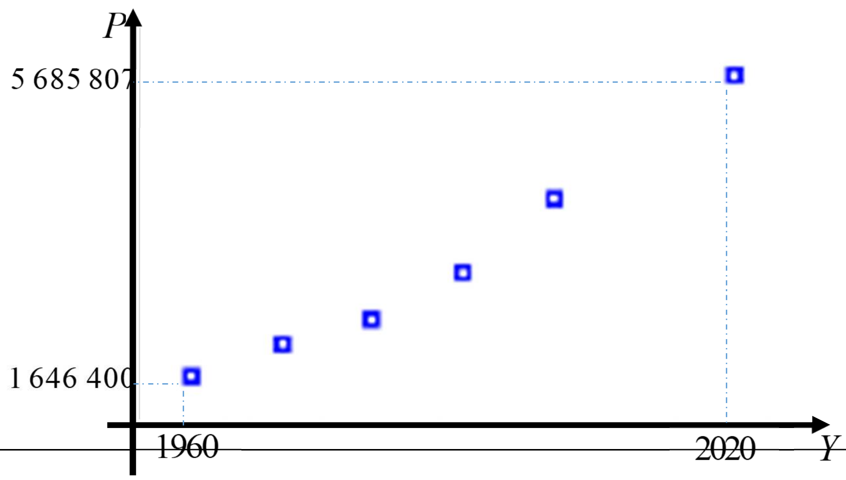
Section B: Probability and Statistics [60 marks]

5 Singapore's total population comprises residents and non-residents. The resident population comprises Singapore citizens and permanent residents. The non-resident population, which excludes tourists and short-term visitors, comprises foreigners who are working, studying or living in Singapore but not granted permanent residence.

The [1]table below shows the total population for Singapore from 1960 to 2020.

Year (Y)	1960	1970	1980	1990	2000	2020
Total population (P)	1,646,400	2,074,507	2,413,945	3,047,132	4,027,887	5,685,807

(i) Sketch a scatter diagram of the data. Use your diagram to explain whether the relationship between Y and P is likely to be well modelled by an equation of the form $P=aY+b$ where a and b are constants. [2]

Solutions	
	

[1] Department of Statistics of Singapore, Title: M810001-Indicators On Population, Annual. https://www.singstat.gov.sg

The linear model $P = aY + b$ is not appropriate because from the scatter diagram, the points do not lie close to a straight line. OR The linear model $P = aY + b$ is not appropriate because from the scatter diagram, P increases at an increasing rate as Y increases.	

(ii) A student wishes to determine whether the relationship between Y and P is modelled better by $\sqrt{P} = aY + b$ or $\ln P = aY + b$.

(a) By calculating the relevant product moment correlation coefficients, correct to 4 significant figures, explain how the student can decide which model is better. [3]

Solutions	
For $\sqrt{P} = aY + b$, product moment correlation coefficient is 0.9942 (4 s.f.) For $\ln P = aY + b$ product moment correlation coefficient is 0.9973 (4 s.f.) Since 0.9973 > 0.9942 which is closer to 1 Hence $\ln P = aY + b$ is a more appropriate model	

(b) Use the model identified in part **(ii)(a)** to estimate the population in 2010. [2]

Solutions	
From GC, $\ln P = 0.0209528974Y - 26.75174675$ $\ln P = 0.0210Y - 26.8$ When Y = 2010 P = 4702368.888 $P = 4.70 \times 10^6$ (to 3 sf)	

6 Tom has a bag of wooden rectangular blocks of identical size. The bag contains 1 blue block, m red blocks and $(m-1)$ yellow blocks, where $m>2$. Tom and Jerry play a game using Tom's bag of wooden blocks. Jerry draws 2 blocks at random, one at a time, without replacement. 3 points will be awarded if a yellow block is drawn, 2 points will be awarded if a red block is drawn, but no points will be awarded if a blue block is drawn. Jerry's final score is the product of the points awarded for the 2 blocks drawn.

Let the random variable X denote Jerry's final score.

(i) Show that $\mathrm{P}(X=0)=\dfrac{1}{m}$ and hence, find the probability distribution of X. [4]

Solutions	
Let X be Jerry's final score: $0,\ 2\times2=4,\ 2\times3=6,\ 3\times3=9$ $\mathrm{P}(X=0)=\mathrm{P}(R,B)+\mathrm{P}(B,R)+\mathrm{P}(Y,B)+\mathrm{P}(B,Y)$ $=\dfrac{m(1)}{2m(2m-1)}\times2+\dfrac{(m-1)(1)}{2m(2m-1)}\times2$ $=\dfrac{4m-2}{2m(2m-1)}$ $=\dfrac{1}{m}$ $\mathrm{P}(X=4)=\mathrm{P}(R,R)$ $=\dfrac{m(m-1)}{2m(2m-1)}$ $=\dfrac{m-1}{2(2m-1)}$ $\mathrm{P}(X=6)=\mathrm{P}(R,Y)+\mathrm{P}(Y,R)$ $=\dfrac{m(m-1)}{2m(2m-1)}\times2$ $=\dfrac{m-1}{2m-1}$ $\mathrm{P}(X=9)=\mathrm{P}(Y,Y)$ $=\dfrac{(m-1)(m-2)}{2m(2m-1)}$	

x	0	4	6	9
$\mathrm{P}(X=x)$	$\dfrac{1}{m}$	$\dfrac{m-1}{2(2m-1)}$	$\dfrac{m-1}{2m-1}$	$\dfrac{(m-1)(m-2)}{2m(2m-1)}$

(ii) Find the value of m if Jerry's expected final score is 5. [2]

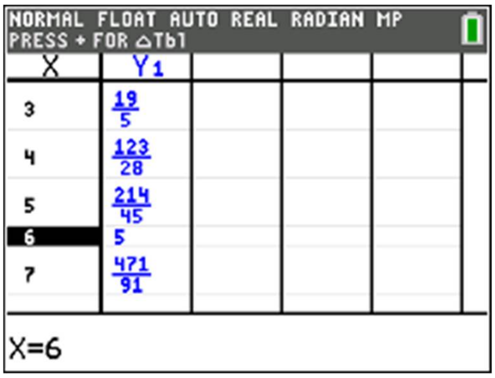
Solutions

$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$

$=\dfrac{4(m-1)}{2(2m-1)}+\dfrac{6m-6}{2m-1}+\dfrac{9(m^2-3m+2)}{2m(2m-1)}$

$=\dfrac{25m^2-43m+18}{2m(2m-1)}$

Using GC, $m=6$ when $\mathrm{E}(X)=5$.



Tom pays Jerry \$5 if Jerry's final score is at least 5 and Jerry pays Tom \$$a$ if his final score is less than 5.

(iii) Using the value of m found in **(ii)**, find the range of values of a if Tom is expected to make a profit. [2]

Solutions

Let W be Tom's winnings (in \$).

When m = 6,

w	a	-5
$\mathrm{P}(W=w)$	$\mathrm{P}(X=0)+\mathrm{P}(X=4)$ $=\dfrac{1}{6}+\dfrac{5}{22}$ $=\dfrac{13}{33}$	$\mathrm{P}(X=6)+\mathrm{P}(X=9)$ $=\dfrac{5}{11}+\dfrac{5}{33}$ $=\dfrac{20}{33}$

If Tom is expected to make a profit, then

$\dfrac{13}{33}a-\dfrac{20(5)}{33}>0$

$13a>100$

$a>7.6923$

Hence, $a>7.70$ (to 2dp) in order for Tom to make a profit.

7 A toy factory manufactures gel beads which are polymer beads that increase in size when soaked in water. On average, 8% of the gel beads are defective. The gel beads are packed in bags of 500. A significant number of customers recently gave feedback that many of the gel beads they bought could not expand in water or cracked while expanding. The quality control department decides to take a random sample of 20 gel beads from each bag to test. If more than 4 gel beads are found to be defective in the sample of 20, the bag is rejected. Otherwise the bag is accepted.

(i) State, in context, two assumptions needed for the number of defective gel beads in the sample to be well modelled by a binomial distribution. [2]

Solutions	
Assumptions • The event that a gel bead is defective is independent of other gel beads. • The probability that a gel bead is defective is a constant at 0.08.	

Assume now that the number of defective gel beads in a sample of 20 is modelled by a binomial distribution.

(ii) Find the probability that a randomly chosen bag of gel beads is rejected. [1]

Solutions	
Let X be the number of gel beads that are defective, out of 20. $X \sim \mathrm{B}(20, 0.08)$ $\mathrm{P}(X \geq 5) = 1 - \mathrm{P}(X \leq 4)$ $= 0.018344$ $= 0.0183 \ (\text{to 3 sf})$	

(iii) An officer from the quality control department is in charge of inspecting 10 randomly chosen bags of gel beads. Find the probability that the last bag inspected is the second bag that is being rejected. [2]

Solutions	
<u>Method 1</u> Required probability $= {}^{9}C_{1} \times 0.018344 \times (1 - 0.018344)^{8} \times 0.018344$ $= 0.00261 \ (\text{to 3sf})$ <u>Method 2</u> Let Y be the number of bags that is being rejected, out of 10. Required probability $= \mathrm{P}(Y = 1) \times \mathrm{P}(\text{10th bag is rejected})$ $= 0.14236 \times 0.018344$ $= 0.00261 \ (\text{to 3sf})$	

(iv) A random sample of 20 gel beads is taken from a particular bag. Given that the bag is rejected, find the probability that there are more than 13 gel beads with no defects in the random sample of 20 gel beads. [3]

Solutions	
$\mathrm{P}(X' > 13 \mid X \geq 5)$ $= \dfrac{\mathrm{P}(X' \geq 14 \cap X \geq 5)}{\mathrm{P}(X \geq 5)}$ $= \dfrac{\mathrm{P}(X \leq 6 \cap X \geq 5)}{\mathrm{P}(X \geq 5)}$ $= \dfrac{\mathrm{P}(5 \leq X \leq 6)}{\mathrm{P}(X \geq 5)}$ $= 0.965245$ $= 0.965$ (to 3 sf)	

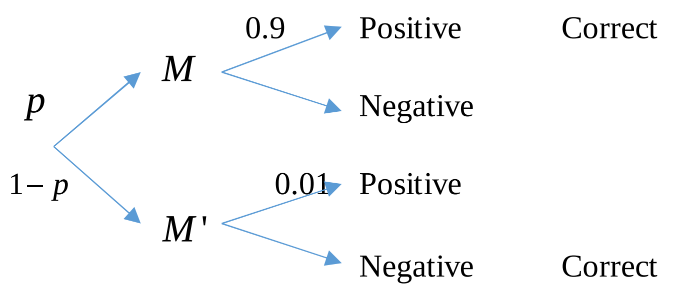
(v) The quality control department now decides to test 50 randomly chosen bags of gel beads. Find the probability that the mean number of defective gel beads found in the sample of each bag will not exceed 1.5. [2]

Solutions	
<u>Method 1</u> $\mathrm{E}(X) = 20 \times 0.08 = 1.6$ $\mathrm{Var}(X) = 20 \times 0.08 \times 0.92 = 1.472$ Since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(1.6, \dfrac{1.472}{50}\right)$ approximately. $\mathrm{P}(\overline{X} \leq 1.5) = 0.280$ (3 s.f.) <u>Method 2</u> Let T be the number of gel beads that are defective, out of 1000. $T \sim \mathrm{B}(1000, 0.08)$ $\mathrm{P}\left(\dfrac{T}{50} \leq 1.5\right) = \mathrm{P}(T \leq 75) = 0.304$ (to 3 sf)	

8 Mr Wong works as the Information Technology manager at a company. To boost the security of the network used by the company, he bought a breach detection system (BDS) which is a defensive tool designed to detect the activity of malware inside a network. The BDS sends out an alert to indicate that a breach has occurred. If there is malicious activity, there is a 90% chance that the BDS will correctly identify the activity as malicious. If the activity is not malicious, there is a 1% chance that the BDS will incorrectly identify the activity as malicious.

It is known that the BDS sends out alerts identifying 109 activities as malicious out of 10000 activities for a particular network system at a particular instance.

(i) Find the probability of an activity being malicious. [2]

Solutions	
 Let the probability of an activity being malicious be p . $0.9p+(1-p)(0.01)=\frac{109}{10000}$ $0.89p=9\times10^{-4}$ $p=0.001011236$ $p=0.00101$ (to 3 sf)	

(ii) Find the probability that the BDS is correct in detecting the type of activity in the network. Give your answer correct to 4 significant figures. [2]

Solutions	
Probability that the BDS is accurate in detecting the type of activity in the network $=0.9p+(1-p)(1-0.01)$ = 0.9899089888 = 0.9899 (to 4 sf)	

(iii) Given that the BDS sends out an alert, find the probability that the activity is not malicious. Give your answer correct to 4 significant figures. [2]

Solutions	
Given that the BDS sends out a positive alert, the probability that the activity is not malicious $= \dfrac{(1-p)(0.01)}{\mathrm{P}(\text{BDS sends out positive alert})}$ $= \dfrac{(1-p)(0.01)}{p\times 0.9+(1-p)\times 0.01}$ = 0.9165034533 = 0.9165 (to 4 sf)	

(iv) Comment on the effectiveness of the BDS in the identification of a security breach. [1]

Solutions	
Since the BDS is able to correctly identify a malicious activity 99.99% of the time which is of very high accuracy given the activity is malicious, the BDS is effective in the identification of a security breach. OR Since there is a 91.65% chance that the activity is not malicious when the BDS sends out an alert, this may result in the IT team having to react to alert unnecessary when the activity is not malicious. Thus the BDS is not very effectiveness in the identification of a security breach.	

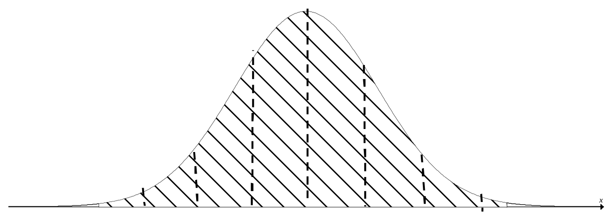
(v) The BDS is able to provide daily reports on the activities on the network which can only be accessible to Mr Wong. During the initial set-up, Mr Wong is required to set a 4-letter code formed from the letters of the word 'NINETEEN' as his password for authentication. Find the number of possible 4-letter codes that Mr Wong can choose from to be his password. [4]

Solutions	
Number of ways when **(a)** All letters distinct = 4! = 24 **(b)** 3 letters are the same = $2\times {}^3C_1\times\dfrac{4!}{3!}=24$ **(c)** 2 letters are the same the other 2 distinct = $2\times {}^3C_2\times\dfrac{4!}{2!}=72$ **(d)** 2 letters are the same = $\dfrac{4!}{2!2!}=6$ Total number of ways = 126	

9 In this question, you should state clearly all the distributions that you use, together with the values of the appropriate parameters.

Mr Tan manages a stall on his own, selling original egglets. Due to limited budget, he can only afford one egglet machine. As such, he can only prepare egglets one at a time. The time needed, in minutes, to make an original egglet follows the distribution $\mathrm{N}(4, 0.25)$.

(i) Sketch the distribution for the time needed to make an original egglet to be between 2 and 6 minutes. [2]

Solutions	
	

(ii) On a particular morning, Mr Tan sold 15 original egglets. Find the probability that there were exactly 6 original egglets which he took less than 4 minutes to make each of them. [2]

Solutions	
Let W be the number of original egglets, out of 15, that Mr Tan took less than 4 minutes to make it. $W \sim \mathrm{B}(15, 0.5)$ $\mathrm{P}(W = 6) = 0.153$ (to 3 sf)	

(iii) Find the probability that the time needed to make 3 randomly chosen original egglets differs from thrice the time needed to make a randomly chosen original egglet by not more than 5 minutes. [3]

Solutions	
Let X be the time needed to make an original egglet. $X \sim \mathrm{N}(4, 0.5^2)$ $X_1 + X_2 + X_3 - 3X \sim \mathrm{N}(0, 3)$ $\mathrm{P}(\lvert X_1 + X_2 + X_3 - 3X \rvert \le 5)$ $= \mathrm{P}(-5 \le X_1 + X_2 + X_3 - 3X \le 5)$	

$= 0.996$	

Hoping to attract more customers, Mr Tan decides to introduce chicken floss egglets and soft drinks at his stall. The time needed, in minutes, to make a chicken floss egglet follows the distribution $\mathrm{N}(6, 1.2)$. The time needed, in minutes, to prepare a cup of soft drink is modelled as 20 % of the time needed to make an original egglet.

(iv) A customer orders an original egglet, a chicken floss egglet and a cup of soft drink. Given that the probability that the total time taken by Mr Tan to prepare these 3 items, one after another, in less than *m* minutes is at most 90%, find the range of possible values of *m*. [4]

Solutions

Let Y be the time needed to make a chicken floss egglet.

$Y \sim \mathrm{N}(6, 1.2)$

Let W be the time needed to prepare a cup of soft drink

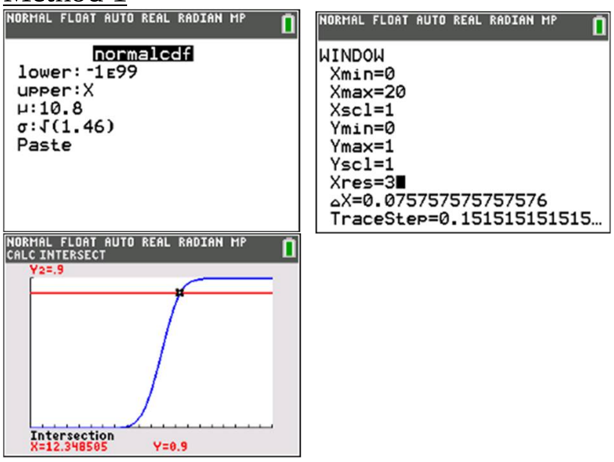
$W = 0.2X \sim \mathrm{N}(0.8, 0.01)$

Let T be total time taken by Mr Tan to prepare the order.

$T = X + Y + W \sim \mathrm{N}(10.8, 1.46)$

$\mathrm{P}(T < m) \le 0.9$

<u>Method 1</u>



$0 < m \le 12.3485$

$0 < m \le 12.3$ (to 3 sf)

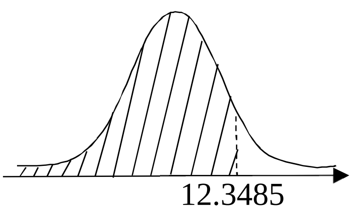
<u>Method 2</u>

Using GC, $\mathrm{P}(T < 12.3485) = 0.9$.

Thus for $\mathrm{P}(T < m) \le 0.9$,

$0 < m \le 12.3485$

$0 < m \le 12.3$ (to 3 sf)



(v) State an assumption needed for your calculations in part **(iv)** to be valid. [1]

Solutions	
The time needed to make an original egglet and the time needed to make a chicken floss egglet is independent of each other. The time needed to make an original egglet is independent of the time needed to prepare a cup of drink. The time needed to make a chicken floss egglet is independent of independent of the time needed to prepare a cup of drink.	

10 Disposable face masks undergo the [2]Bacterial Filtration Efficiency (BFE) Test to assess how well a mask filters droplets containing biological agents such as bacteria or viruses. It is known that the higher the BFE, the more effective a mask is in preventing bacteria-containing droplets from reaching the wearer. For instance, a mask with a BFE of 95% will meet the requirements for medical and surgical masks since it blocks 95% of droplets it is exposed to.

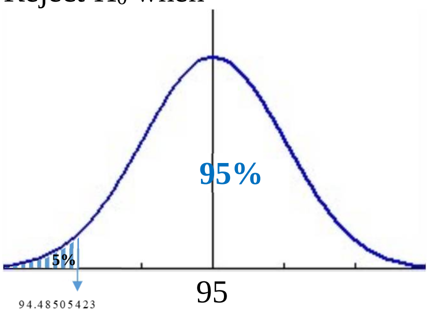
A company that manufactures disposable face masks claimed that their masks are rated with BFE of at least 95%. An intern from the company wishes to check if the claim made by the company is valid. He is told that the BFE of the disposable face masks manufactured is distributed normally and that the standard deviation is 0.99%. He decides to carry out a hypothesis test at 5% level of significance on a random sample of 10 disposable face masks.

(i) Explain what is meant by a random sample in this context. [1]

Solutions	
A random sample is obtained such that • Every mask manufactured by the company **has an equal chance of being chosen** and • The event of one mask being chosen is **independent** of the event of another mask being chosen.	

[2] A-STAR Explainer: Testing the Efficacy of Protective Face Masks.*https://www.a-star.edu.sg/News-and-Events/a-star-news/news/covid-19/*

(ii) State the hypotheses for the test, defining any symbols that you used, and find the set of possible mean BFE of the 10 randomly chosen disposable masks corresponding to the critical region. [4]

Solutions	
Let X (in %) be the BFE of a mask, μ be the population mean BFE and σ^2 be the population variance $\text{H}_0 : \mu = 95$ $\text{H}_1 : \mu < 95$ $n = 10$ and $\sigma^2 = 0.99^2$ <u>Method 1</u> $\bar{X} \sim \text{N}\left(95, \dfrac{0.99^2}{10}\right)$ Level of significance = 5% Reject H_0 when  $\bar{x} \le 94.48505423$ $\bar{x} \le 94.4$ (to 3 sf) $\therefore \{\bar{x} \in \mathbb{R} : 0 < \bar{x} \le 94.4\}$ <u>Method 2</u> Under H_0, $\bar{X} \sim \text{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$. Test statistic $Z = \dfrac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \text{N}(0,1)$ Level of significance = 5% Reject H_0 when $z\text{-value} \le -1.64485$ $\dfrac{\bar{x} - 95}{\frac{0.99}{\sqrt{10}}} \le -1.64485$ $\bar{x} \le 94.48505423$ $\bar{x} \le 94.4$ (to 3 sf)	

$\therefore\{\bar{x}\in\mathbb{R}:0<\bar{x}\le 94.4\}$	

(iii) Given that the null hypothesis is rejected in the test conducted by the intern at 5% level of significance, comment if the same conclusion is obtained when the test is conducted at 1% level of significance. [1]

Solutions	
Since H_0 is rejected at 5% significance level, then p-value ≤ 0.05. If p-value ≤ 0.01, then H_0 is rejected at 1% significance level so the conclusion remains the same. If $0.01 < p$-value ≤ 0.05, then H_0 is not rejected at 1% significance level so the conclusion has changed. Thus the same conclusion may or may not be obtained.	

The company modified the manufacturing process to increase the BFE of the masks produced to be more than 98%. A quality control manager decides to perform a hypothesis test on a random sample of 70 disposable face masks produced by the modified manufacturing process to find out if this is the case.

The distribution of the BFE, y %, of the random sample of 70 disposable face masks produced by the modified manufacturing process are given as follows.

BFE, y %	95.8	96.8	97.5	98.3	98.5	98.7	98.8	99.2	99.5	99.8
Number of disposable face masks	3	8	11	10	10	9	5	5	7	2

(iv) Explain why the quality control manager takes a sample of 70 new masks but the intern only takes a sample of 10 masks. [2]

Solutions	
For the intern, the BFE of each disposable face mask is known to be normally distributed. However the BFE of the new batch of disposable face masks is **not known to be normally distributed**. Hence, the quality control manager needs to **take a large sample** so that the **sample mean BFE can be approximated to follow a normal distribution** by **Central Limit Theorem**.	

(v) Carry out a hypothesis test at 5% level of significance for the quality control manager. Give your conclusion in context. [4]

Solutions

Let Y (in %) be the BFE of a mask manufactured under the modified process,
μ be the population mean BFE of the masks manufactured under the modified process and
σ^2 be the population variance of Y.

$\mathrm{H}_0 : \mu = 98$

$\mathrm{H}_1 : \mu > 98$

Under H_0, since sample size = 70 is large, by **Central Limit Theorem**, $\overline{Y} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ **approximately**.

Test Statistic $Z = \dfrac{\overline{Y} - \mu}{\sqrt{\dfrac{S^2}{n}}} \sim \mathrm{N}(0,1)$ approximately.

Level of Significance = 5%

From GC, p-value = $0.020053057 \approx 0.0201(\text{to 3 sf})$

Since p-value = $0.0201 < 0.05$, then we reject H_0 and conclude at 5% level of significance that there is sufficient evidence for the quality control manager to say that the population mean BFE of the disposable face masks manufactured by the new process is more than 98%.

GC Keystrokes: