Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Theorem

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2}bc \sin A$$

1 **(i)** Sketch the graph of $y = -|3x-5|$, indicating the intercepts clearly. [2]

(ii) Explain why the equation $-2 = |3x-5|$ has no real roots using a graphical approach. [1]

2 It is known that $\log_2 b = y$ and $\log_b y = 3$.

(i) Find an expression for $\log_{\sqrt{2}} b - \log_2 2b$ in terms of y.

(ii) By considering a pair of simultaneous equations, show that $\lg b = b^3 \lg 2$.

3 (i) On the same diagram, sketch the graphs of $\frac{y^2}{25} = x$ and $y = -3x^{\frac{3}{2}}$. [3]

(ii) Find the coordinates of the points of intersection of the two curves. [3]

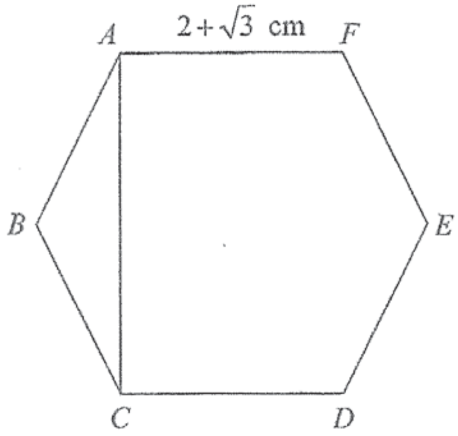
4 **(i)** Factorise $8x^3 + 27$. [

(ii) Express $\dfrac{8x^3 + 27}{(2x^2 + 3x)(x-1)^2}$ in partial fractions. [

5 **(i)** Given that $u = 2^{2y}$, solve the equation $8^{y} = 7(2^{y}) + \dfrac{8}{2^{y}}$. [4]

(ii) Hence, solve the equation $64^{-x} = 7(4^{-x}) + \dfrac{8}{4^{-x}}$. [2]

6 A regular hexagon $ABCDEF$ with sides $\left(2+\sqrt{3}\right)$ cm is shown below.



(i) Show that $AC = \left(2\sqrt{3}+3\right)$ cm.

(ii) If the line segment AC has length $\left(\sqrt{27}-\sqrt{3}\right)r$ cm, find the value of r, expressing answer in the form $a+b\sqrt{3}$, where a and b are rational numbers.

7 The roots of $x^2 - 7x + 4 = 0$ are α^2 and β^2. Given that α and β are opposite in sign,

(i) Find two possible values of $\alpha + \beta$. [4]

(ii) Find two non-equivalent quadratic equations whose roots are α and β. [2]

8 (i) Show that $\frac{d}{dx}(2x\sin^2 x) = 2\sin^2 x + 4x\sin x\cos x$.

(ii) Hence find $\int x\sin 2x \, dx$.

9 (i) Using $\tan 3x = \tan(2x + x)$, show that $\tan 3x$ may be expressed as $\dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$. [3]

(ii) Find all the values of x between 0 and π for which $\tan 3x = -5\tan x$. [5]

10 (i) Show that the curve $y=\frac{1}{4}x^2-hx+4$ meets the line $y=2x-(2h-1)$ for any real val

(ii) State the value of h if the line $y=2x-(2h-1)$ is a tangent to the curve $y=\frac{1}{4}x^2-hx$

(iii) Using your answer from part (ii), find the values of p and q if the line $y=2x$- intersects the line $y=px+q$ at an infinite number of points.

11 A semi-circle has radius r m, area A m^2 and perimeter P m. At the instant when its radius is a m, its area is increasing at a rate of 2π m^2/s.

(i) Find an expression in terms of a, for the rate of increase of the radius at this instant. [3]

(ii) Find an expression in terms of a and π, for the rate of increase of the perimeter at the same instant. [2]

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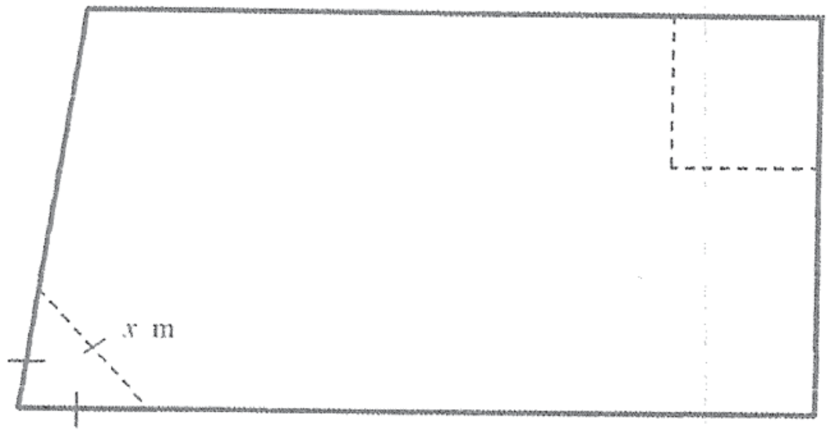


The diagram shows a room (crime scene) surrounded by solid walls. Two bodies are found at extreme corners of the room. To facilitate forensic work, a tape of total length p m is used to off these two extreme corners. The dotted lines represent the length of tape used. One corne room is an equilateral triangle of side x m and the other corner is a square.

(i) Show that the area of the triangle is $\frac{\sqrt{3}}{4}x^2$.

(ii) Show that the total area, A m^2, of the two cordoned-off corners is given by

$$A = \frac{p^2}{4} - \frac{p}{2}x + \frac{\sqrt{3}+1}{4}x^2.$$

(iii) Given that x can vary depending on how taut the tape is, find an expression for x in terms of p for which the area A is stationary. [2]

(iv) Write down, but do not simplify, an expression for the stationary value of A in terms of p and determine the nature of this stationary value. [3]

(v) In the case where $p = \sqrt{3} + 1$, sketch a graph of A against x, indicating the turning point and the y-intercept clearly. [3]

End of Paper 1

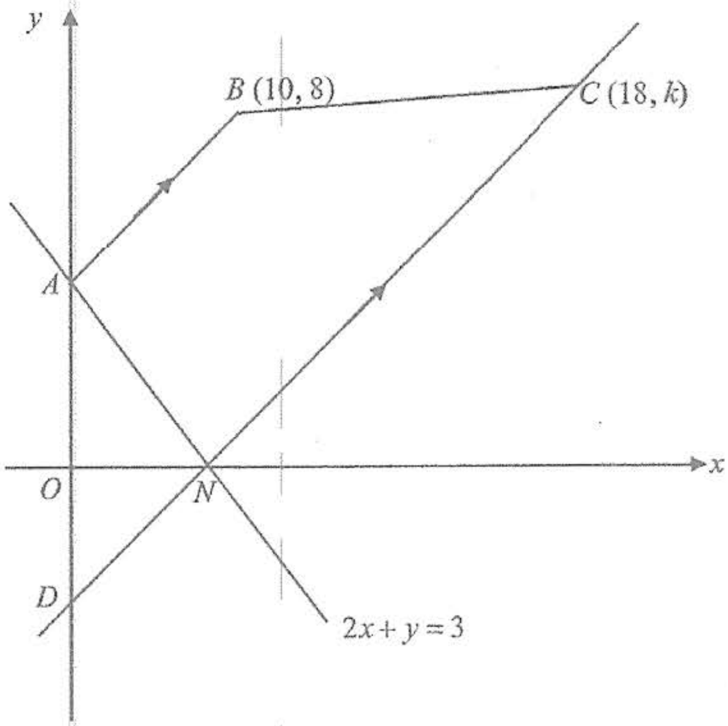
Paper 2

1 (i) Find the remainder when the polynomial $f(x) = 4x^3 - x + 3$ is divided by $2x - 1$. [

(ii) Given that $f(x) - 5 = (2x-1)A(x) + b$, where $A(x)$ is a polynomial, state the value of b the degree of $A(x)$. [

(iii) A constant c is added to $f(x)$ to make it divisible by $2x - 1$. State the value of c. [

2 **Solutions to this qu[obscured] by accurate drawing will not be accepted.**



The diagram shows a trapezium $ABCD$ in which AB is parallel to DC. The coordinates of the points B and C are (10, 8) and (18, k) respectively. The line with equation $2x + y = 3$ intersects the x-axis and y-axis at N and A respectively. The line CN intersects the y-axis at D.

(i) Determine whether $\angle ANC = 90^\circ$. [4]

(ii) Write down the equation of the line CD.

(iii) Find the value of k and hence, find the area of the trapezium $ABCD$.

3 The curve $y = f(x)$ is such that $f''(x) = x^2 - \frac{1}{2}$.

(i) Find the range of values of x for which $f'(x)$ is a decreasing function. [3]

The point $P(3, 10)$ lies on the curve. The gradient of the curve at P is $\frac{5}{2}$.

(ii) Find the equation of the curve. [5]

4 (i) Given that the coefficient of the $\frac{1}{x^2}$ term in the binomial expansion of $\left(\frac{1}{x^3}-kx^2\right)^9$ is 4032, show that $k=-2$.

(ii) Find the constant term in the expansion of $\left(x^{12}+x^7\right)\left(\frac{1}{x^3}-kx^2\right)^9$.

5 A curve C has equation $y = \ln(1-2x)$, $x < \frac{1}{2}$. The point P on C has coordinates $\left(\frac{1}{3}, \ln\frac{1}{3}\right)$. The tangent and normal to C at P meets the y-axis at Q and R respectively.

(i) Find the equations of the tangent and normal to the curve C at P. [4]

(ii) Find the coordinates of Q and of R [2]

(ii) Find the area of triangle OPR. [2]

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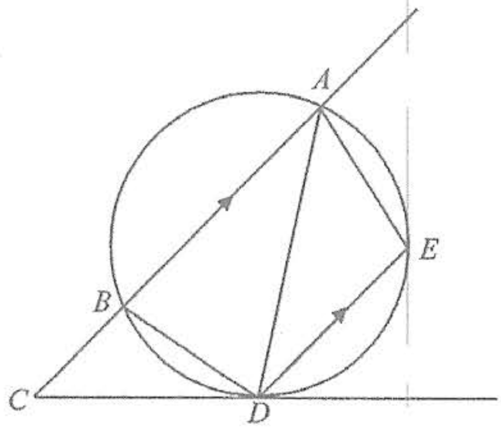

In the diagram, *ABDE* is a cyclic quadrilateral in which *BA* is parallel to *DE*. The tang circle at *D* meets *AB* produced at *C*. The chord *AD* bisects angle *BAE*.

(i) Prove that angle $BCD = \frac{1}{2}$ angle *BAE*.

(ii) If *AD* is the diameter of the circle,
(a) prove that ΔABD is similar to ΔDEA,

(b) state the name of the geometric shape given to the quadrilateral *ABDE*.

7 A boy runs along the coastline of a beach and passes a fixed point A. The velocity, v m/s, that he runs in t seconds after he passes A is given by

$$v = 20e^{-0.9t} - 3.$$

(i) Find the distance that the boy ran 60 seconds after he passes A. [5]

(ii) Find the boy's acceleration when he is instantaneously at rest. [4]

(iii) Explain what the sign of the acceleration indicates. [1]

(iv) Explain whether the boy will be running at his maximum velocity at any point of time of his run. [2]

The function $f(x) = 2x^3 + bx^2 - 14x - 20$ has a factor of $a(x+5)(x+1)$, where a and b are positive constants.

(i) Find the value of b. [2]

(ii) If $(x-2)$ is also a factor of $f(x)$, state the value of a. [1]

(iii) Hence, deduce another function $g(x)$ whose coefficient of x^3 is 3 and values of the roots of $g(x) = 0$ are twice the values of the roots of $f(x) = 0$. [1]

9 The function $f(x) = a\cos bx + 2$ has a period of 6π and a range of $-2 \le f(x) \le 6$.

(i) State the value of a and of b.

(ii) Sketch the graph of $y = f(x)$ for $0 \le x \le 9\pi$.

(iii) On the same diagram, sketch $g(x)$ where $g(x) = -f(x)$ for $0 \le x \le 9\pi$.

(iv) State the equation of the line of symmetry between $f(x)$ and $g(x)$.

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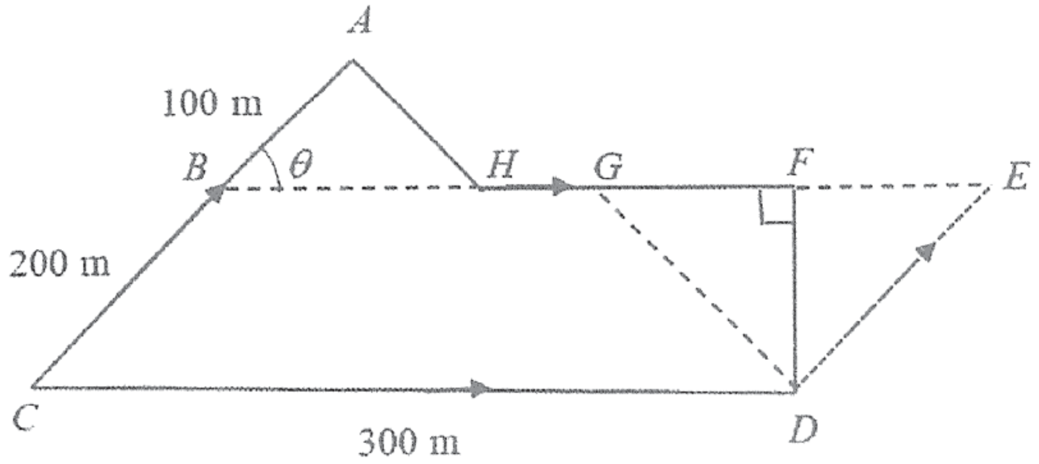


The diagram shows the running path ($A \rightarrow B \rightarrow C \rightarrow D \rightarrow F \rightarrow G \rightarrow H \rightarrow A$) of Ali. ΔABH and ΔDEG are isosceles triangles. CA is parallel to DE and CD is parallel to HF. $AB = 100$ m, $BC = 200$ m and $CD = 300$ m. It is also given that angle $GFD = 90°$ and angle $ABH = \theta°$, where $0° < \theta < 90°$.

(i) Given that Ali runs at a uniform speed of 10 m/s throughout, show that the time taken t s for the run can be expressed as $100 + 20\sin\theta - 40\cos\theta$. [5]

(ii) Express t in the form of $100 + R\sin(\theta - \alpha)$, where $R > 0$ and α is an acute angle. [3]

(iii) Using your answer from part **(ii)**, justify with working whether Ali can complete his run in the shortest possible time, assuming that Ali completes his run. [2]

A circle, C_1, has equation $x^2 + y^2 - 4x + 6y = 12$.

(i) Find the radius and coordinates of the centre of the circle C_1. [3]

(ii) Determine whether $(3, 1)$ lies inside, outside or on the circle. [2]

Another circle, C_2, has centre $(2, -8)$ and the same radius as C_1.

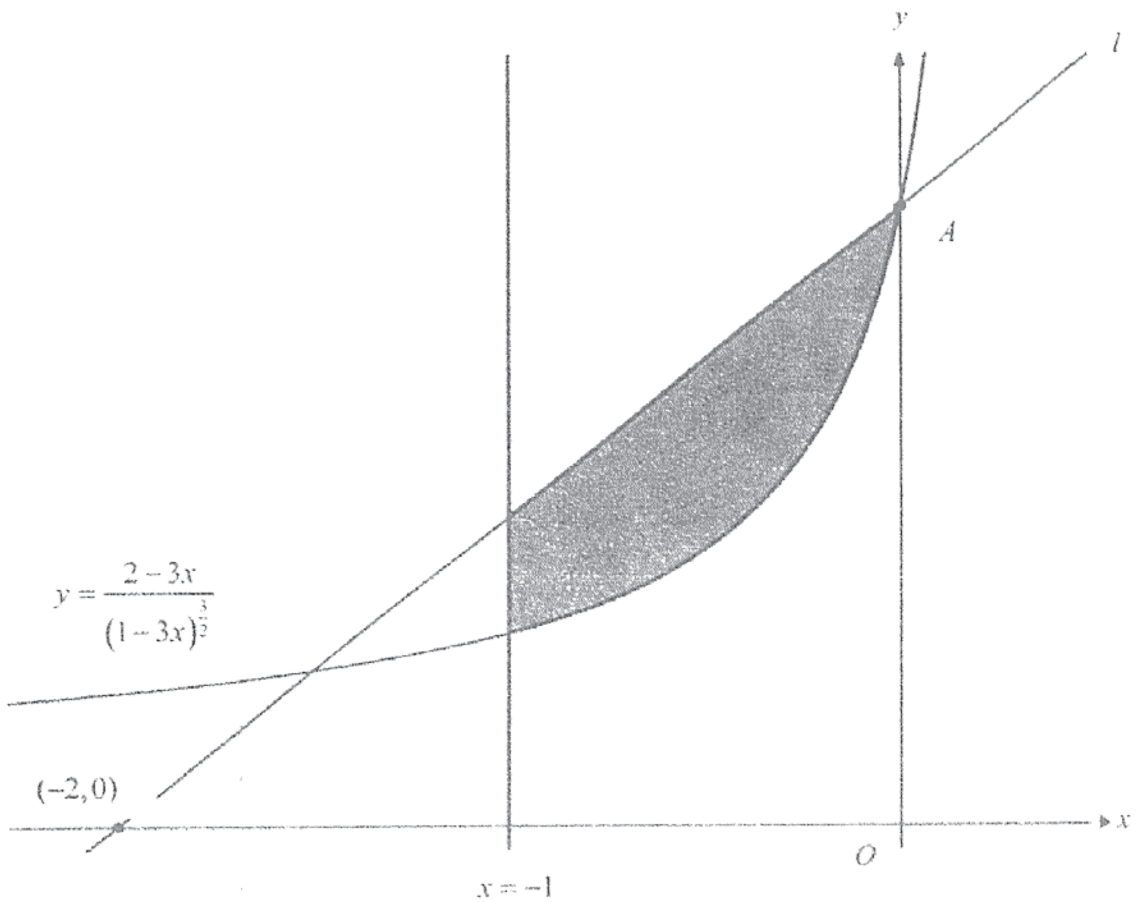
(iii) State the equation of C_2. [1]

The line which is parallel to the x-axis and passes through $(2, -8)$ intersects C_2 at A and B.

(iv) State the equations of the tangents to C_2 at A and B. [2]

12 (a) Show that $\dfrac{d}{dx}\left(\dfrac{2x}{\sqrt{1-3x}}\right)=\dfrac{2-3x}{(1-3x)^{\frac{3}{2}}}$.

(b)



The diagram shows the line $x = -1$ and part of the curve $y = \frac{2-3x}{(1-3x)^{\frac{3}{2}}}$. The curve intersects the y-axis at point A. The line l through A intersects the x-axis at $(-2, 0)$.

(i) Determine the area of the shaded region bounded by the curve, the line $x = -1$ and the line l. [4]

(ii) The area bounded by the x-axis, the curve, the line $x = -1$ and the line $x = a$ is four times the area of the shaded region in part (i), where $a > 0$. Find the value of a. [3]

3 The population, P, in millions of a country on 1st January has been increasing every year from 1960 to 2000. The increase is exponential and so can be modelled by an equation of the form

$$P = P_0 e^{kt} ,$$

where P_0 and k are constants and t is the time in years since 1st January 1960. The table below gives values of P and t for some of the years 1960 to 2000.

Year	1960	1970	1980	1990	2000
t years	0	10	20	30	40
P	5.75	9.97	17.3	30.0	51.9

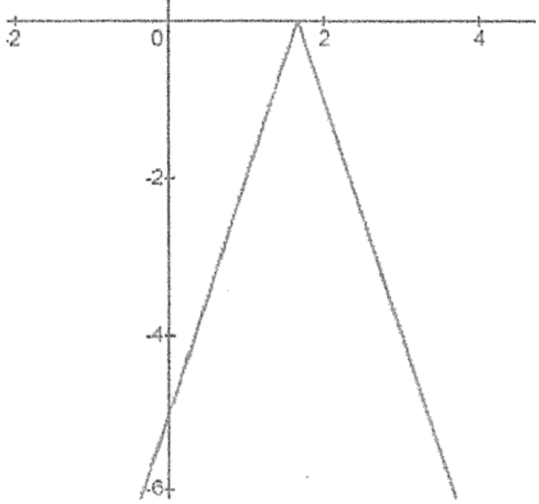
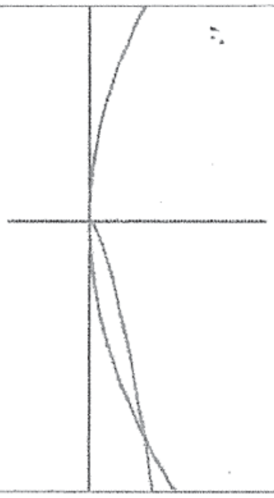
(i) Plot a suitable straight line graph to show that the model is valid for years 1960 to 2000. [3]

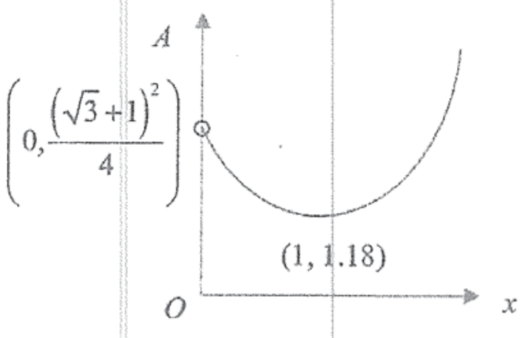
(ii) Estimate the value of k. [2]

(iii) Assuming that the model is still appropriate, use your graph to estimate the population in 1st January 2005. [2]

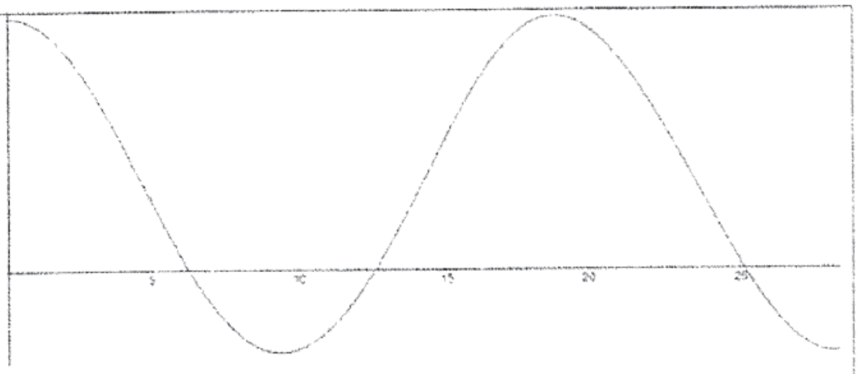
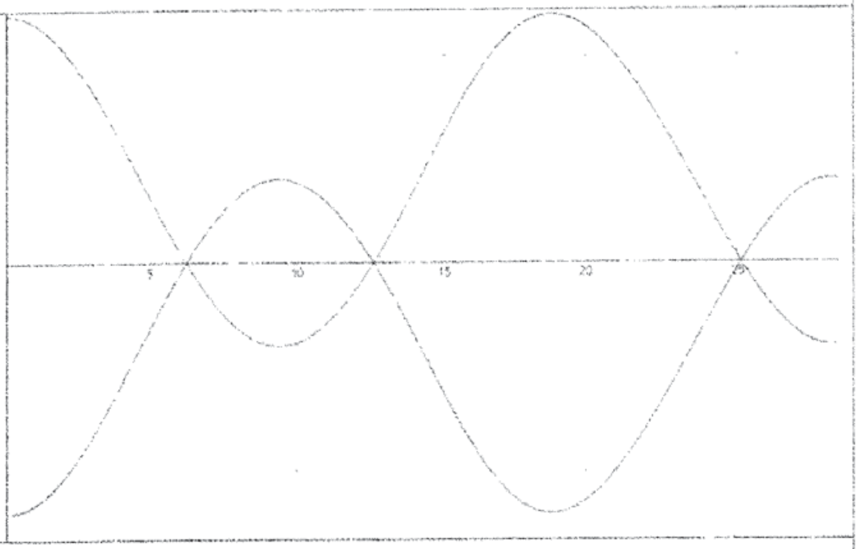
End of Paper 2

2019 4E5N PRELIM AM P1 ANSWER KEY

Q	Solution
1i	
1ii	Since the line y = 2 does not intersect the graph, there are no real roots for the equation $-2 = \|3x-5\|$.
2i	$y-1$
3i	
3ii	$(0,0)$ and $(\frac{5}{3}, -6.45)$
4i	$8x^3+27=(2x+3)(4x^2-6x+9)$
4ii	$\frac{8x^3+27}{(2x^2+3x)(x-1)^2}=\frac{9}{x}-\frac{5}{x-1}+\frac{7}{(x-1)^2}$
5i	$y=\frac{3}{2}$
5ii	$x=-\frac{3}{4}$
6i	$AC=2\sqrt{3}+3$
6ii	$r=1+\frac{1}{2}\sqrt{3}$
7i	$\alpha+\beta=\sqrt{3}$ or $\alpha+\beta=-\sqrt{3}$

7ii	$x^2-\sqrt{3}x-2=0$ or $x^2+\sqrt{3}x-2=0$
8i	$2\sin^2 x+4x(\sin x)(\cos x)$
8ii	$\int x(\sin 2x)\mathrm{d}x = x\sin^2 x-\frac{x}{2}+\frac{1}{4}\sin 2x+C_3$
9i	$\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$
9ii	$x=0.615,\ 2.53$
10i	Since $(h+1)^2\geq 0$ for all values of h, there will always intersections between the line and curve, hence $y=\frac{1}{4}x^2-hx+4$ and $y=2x-(2h-1)$ meet.
10ii	$h=-1$
10iii	$p=2$ $q=3$
11i	$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{2}{a}$
11ii	$\frac{4+2\pi}{a}$
12i	$\left(\frac{\sqrt{3}}{4}\right)x^2$
12ii	$\frac{p^2}{4}-\frac{p}{2}x+\frac{\sqrt{3}+1}{4}x^2$
12iii	$x=\frac{p}{\sqrt{3}+1}$
12iv	$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=\frac{\sqrt{3}+1}{2}>0$ $\therefore A$ is minimum
12v	When $p=\sqrt{3}+1$, $x=1$ $A=1.18$ 

Qns	Answer Key
1i	3
1ii	$b=-2$ Degree = 2
1iii	$c=-3$
2i	$m_{AN}\times m_{CN}=-2\times\frac{1}{2}=-1$, therefore $\angle ANC=90^{\circ}$
2ii	Equation of line *CD* $y=\frac{1}{2}x-\frac{3}{4}$
2iii	52.5 units^2
3i	$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$ (Accept: $-0.707<x<0.707$)
3ii	$f(x)=\frac{x^4}{12}-\frac{1}{4}x^2-5x+20.5$
4i	$r=5$ $k=-2$
4ii	2688
5i	$y=\frac{1}{6}x-\frac{1}{18}+\ln\frac{1}{3}$
5ii	$R(0,\ \ln\frac{1}{3}-\frac{1}{18})$ [Accept: R (0, -1.15)
5iii	0.192 units^2 (3 s.f)
6iib	Rectangle
7i	Distance = 158m
7ii	-2.70 m/s^2
7iii	It means that his velocity is decreasing/ boy is slowing down
7iv	$\frac{dv}{dt}=-18e^{-0.9t}$ For $t\geq 0$, $-18<0$ and $e^{-0.9t}>0$, $\therefore \frac{dv}{dt}=-18e^{-0.9t}<0, \therefore \frac{dv}{dt}\neq 0$. Hence, the boy will be **NOT** running at his maximum velocity at any point of time of his run.
8i	$b=8$
8ii	$f(x)=2x^3+8x^2-14x-20=a(x+5)(x+1)(x-2)$ $\therefore a=2$
8iii	$g(x)=3(x+10)(x+2)(x-4)$
9i	$a=4,\ b=\frac{1}{3}$

9ii	
9iii	
9iv	$y = 0$
10i	$100 + 20\sin\theta - 40\cos\theta$
10ii	$t = 100 + \sqrt{2000}\sin(\theta - 63.4^\circ)$
10iii	Since no θ within the range of $0^\circ < \theta < 90^\circ$ can be found, Ali cannot complete his run in the shortest possible time.
11i	Radius $r = \sqrt{(-2)^2 + (3)^2 - (-12)} = 5$
11ii	Distance of $(3,1)$ to centre $= \sqrt{(3-2)^2 + (1+3)^2} = \sqrt{17} < 5$ Since distance of $(3,1)$ to centre is smaller than the radius of the circle, $(3,1)$ lies inside the circle.
11iii	$(x-2)^2 + (y+8)^2 = 25$
11iv	$x = -3, x = 7$
12a	$\dfrac{2-3x}{(1-3x)^{\frac{3}{2}}}$
12bi	$\dfrac{1}{2}$ square units
12bii	$a = 0.25$ or $a = -1$ (rejected since $a > 0$)
13ii	$k = 0.055$ (Allow $0.05 \le k \le 0.06$)
13iii	Population is 67.4 **millions**