	Class	Index Number
Candidate Name ______________________		

ST ANDREW'S SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2019
SECONDARY 4 EXPRESS & 5 NORMAL ACADEMIC

ADDITIONAL MATHEMATICS — 4047/01
Paper 1

WEDNESDAY **28 August 2019** **2 hours**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a pencil for any diagrams or graphs
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Write your answers on the separate writing paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is 80.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

This question paper consists of 15 printed pages.

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$

2. TRIGONOMETRY

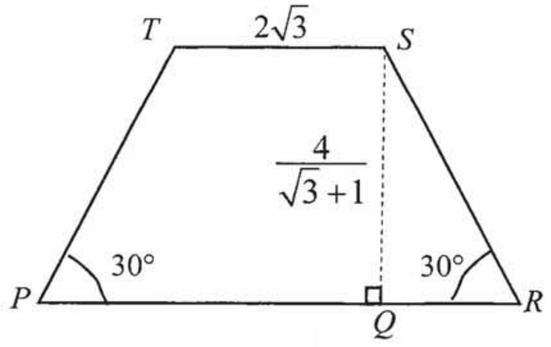
Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

1 In the diagram below, $PQRST$ is a trapezium where angle QRS = angle $TPR = 30°$. SQ is the height of the trapezium and the length of SQ is $\frac{4}{\sqrt{3}+1}$ cm. The length of TS is $2\sqrt{3}$ cm. Find the area of the trapezium $PQRST$ in the form $\left(a\sqrt{3}-12\right)\text{cm}^2$, where a is an integer. [5]

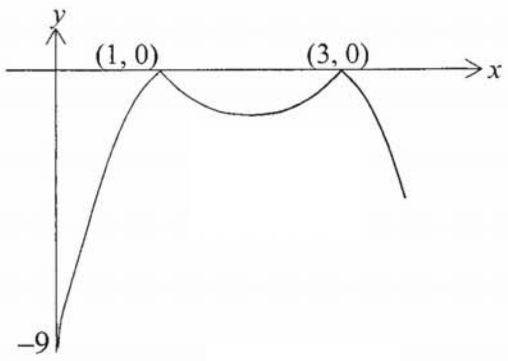


2 Express $\dfrac{4x^3 + x^2 + 6}{(x-2)\left(x^2+2\right)}$ in partial fractions. [5]

3 **(a)** Factorise $8x^3-(x-1)^3$ completely. [3]

(b) Without finding the solution, explain why the equation $8x^3-(x-1)^3=0$ has only one real root. [2]

4 The diagram shows part of the curve $y = -\left|a(x-h)^2 + k\right|$, where $a > 0$. The curve touches the x-axis at (1, 0) and (3, 0) and has a minimum point at (h, k). The curve also cuts the y-axis at -9.

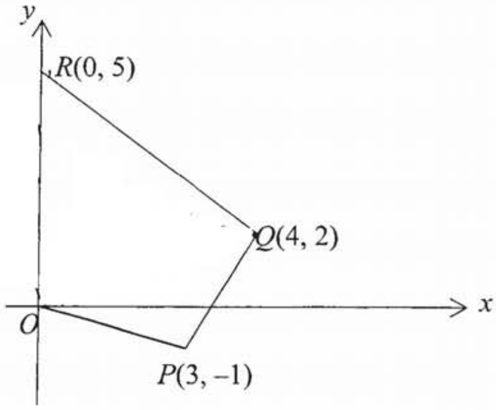


(i) Explain why $h = 2$. [1]

(ii) Determine the value of a and of k. [3]

(iii) Find the set of values of m for which the line $y = mx$ intersects the curve at four distinct points. [2]

5 In the diagram, the coordinates of P, Q and R are $(3, -1)$, $(4, 2)$ and $(0, 5)$ respectively.



(i) Find the equation of the perpendicular bisector of OQ. [3]

(ii) Name the quadrilateral $OPQR$. Justify your answer. [2]

(iii) Given that T is a point on PR such that $OPQT$ is a rhombus, find the coordinates of T. [2]

6 The roots of the quadratic equation $4x^2 + px + q = 0$ are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(i) Given that $\alpha + \beta = 5$ and $\alpha\beta = 2$, find the value of p and q.

(ii) Find the quadratic equation whose roots are $\frac{2\alpha^2}{\beta}$ and $\frac{2\beta^2}{\alpha}$. [5]

7 **(a)** Find the range of values of a for which $x^2 + ax + 2(a-1)$ is greater than 1. [3]

(b) The equation of a curve is $y = 3x^2 + 4x + 6$.

(i) Find the set of values of x for which the curve is above the line $y = 6$.

(ii) Show that the line $y = -8x - 6$ is a tangent to the curve. [2]

8 **(a)** Find the minimum gradient of $y = 2x^3 - 9x^2 - 1$. [3]

(b) The curve $y = x^3 - 6x^2 + k$ touches the positive x-axis at point A.

(i) Find the coordinates of point A. [2]

(ii) Find the value of k

(iii) Find the value of $\frac{d^2y}{dx^2}$ at A and hence the nature of this point. [2]

9 **(a)** Show that $\dfrac{5-10\cos^2 A}{\sin A - \cos A}$ can be written as $k(\sin A + \cos A)$ and state the value of k.

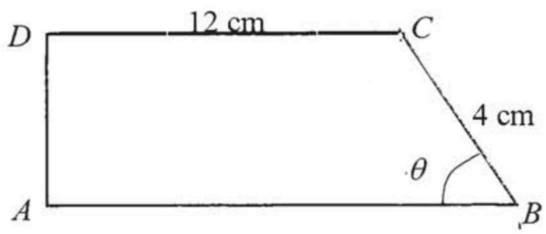
(b) Given that $\sin A = -p$ and $\cos B = -q$, where A and B are in the same quadrant and p and q are positive constants, find the value of

(i) $\sin(-A)$, [1]

(ii) $\tan(45° - A)$,

(iii) $\sec(2B)$. [2]

10 The diagram shows a trapezium $ABCD$ in which $CD = 12$ cm, $BC = 4$ cm and angle $ABC = \theta$ radians, where θ is acute.



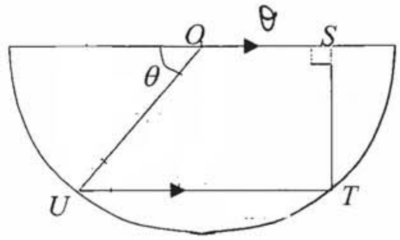
(i) Show that the area, A cm^2, of the trapezium $ABCD$ is given by

$$A = 48\sin\theta + 4\sin 2\theta.$$ [3]

(ii) Given that θ can vary, find the value of θ for which the area of the trapezium A is maximum. [5]

(iii) Hence find the maximum value of A. [1]

11



The diagram shows a trapezium $OSTU$ inscribed in a semi-circle of centre O and radius 10 cm. OU makes an angle θ with the diameter. UT is parallel to the diameter and ST is perpendicular to the OS. The perimeter of the trapezium is L cm.

(i) Show that $L = 10 + 30\cos\theta + 10\sin\theta$. [3]

(ii) Express L in the form $a + R\cos(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. [3]

(iii) Anthony claims that the perimeter of $OSTU$ is 50 cm. Is his claim reasonable? Justify your answer. [2]

(iv) Find the value of θ for which $L = 35$. [2]

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ST ANDREW'S SECONDARY SCHOOL
PRELIMINARY EXAMINATION 2019
SECONDARY 4 EXPRESS & 5 NORMAL ACADEMIC

ADDITIONAL MATHEMATICS — 4047/02
Paper 2

THURSDAY **29 August 2019** **2 hours 30 minutes**

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a pencil for any diagrams or graphs
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** questions.
Write your answers on the separate writing paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
At the end of examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total of the marks for this paper is 100.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

This question paper consists of 20 printed pages.

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{r} a^{n-r} b^r + \cdots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{(n-r)!r!} = \dfrac{n(n-1)\cdots(n-r+1)}{r!}$

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$$\sin^2 A + \cos^2 A = 1$$
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$$\sin 2A = 2 \sin A \cos A$$
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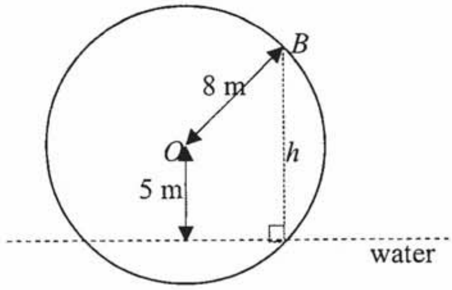
1 A curve is such that $\frac{d^2y}{dx^2} = 6x - 2$ and $P(2, -8)$ is a point on the curve. The gradient of the normal at P is $-\frac{1}{2}$. Find the equation of the curve. [6]

2 **(i)** On the same axes, sketch the graphs $y = \sqrt{288x}$ and $y = 3x^3$ for $x > 0$. [2]

(ii) The tangent to the curve $y = 3x^3$ at point A is parallel to the line passing through the two points of intersection of the curves drawn in **(i)**.
Find the x-coordinate of A. [4]

3



A waterwheel rotates 5 revolutions anticlockwise in 1 minute. Tom starts a stopwatch when the bucket B is at its highest height above water level. The radius of the waterwheel is 8 m and its centre is 5 m above the water level.

The height of bucket B above water level is given by $h = a \cos bt + c$, where t is the time, in seconds, since Tom started the stopwatch.

(i) Determine the value of each of the constant a, b and c. [3]

(ii) For how long is $h < 0$? [3]

4 In the binomial expansion of $x\left(2x+\frac{k}{x}\right)^8$, where k is a positive constant, the coefficient of x^3 is 28.

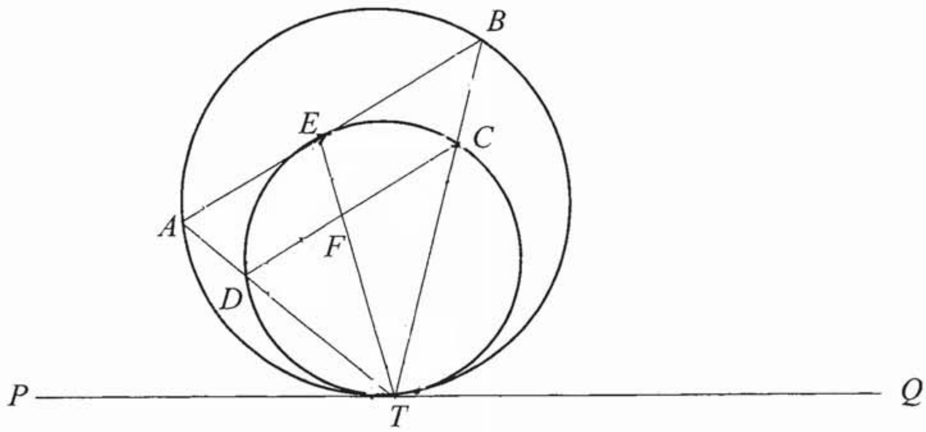
(i) Show that $k=\frac{1}{4}$. [4]

(ii) Explain why there is no constant term in the expansion of $x\left(2x+\frac{k}{x}\right)^8$. [1]

(iii) Hence find the coefficient of x^3 in the expansion of

$$x\left(2x+\frac{k}{x}\right)^8+k\left(1-x\right)^{10}.$$ [2]

5



In the diagram, the two circles touch each other at T and PTQ is their common tangent. AB is a tangent to the smaller circle at E. AT and BT cut the smaller circle at D and C respectively. ET and CD intersect at F. Prove that

(i) AB is parallel to DC, [2]

(ii) the line TE bisects angle ATB, [3]

(iii) triangle DFT is congruent to triangle EFC if $DF = EF$. [2]

6 The variables x and y are related by an equation of the form $y - x = \frac{b}{a}x^2 + b$.

Corresponding values of x and y are shown in the table below.

x	1	2	3	4
y	2.73	7.5	14.75	24.5

(i) Using suitable variables, draw on the graph paper, a straight line graph. [3]

(ii) Using the graph, estimate the value of each of constants a and b. [2]

(ii) By drawing a suitable straight line on your graph, estimate the value of x and y when $y = \frac{1}{2}x^2 + x + 2$. [2]

7 The term containing the highest power of x and the term independent of x in the polynomial $\mathrm{P}(x)$ are $2x^4$ and -3 respectively. It is given that $\left(2x^2+x-1\right)$ is a quadratic factor of $\mathrm{P}(x)$ and the remainder when $\mathrm{P}(x)$ is divided by $(x-1)$ is 4.

(i) Find the polynomial $\mathrm{P}(x)$ and factorise it completely. [4]

(ii) Solve $\mathrm{P}(x) = 0$. [1]

(iii) Find the values of x that satisfy the equation $\mathrm{P}(1-x) = 0$. [2]

8 **(a)** Given that $y = x^2 e^{3x}$, find the range of values of x for which y is an increasing function. [4]

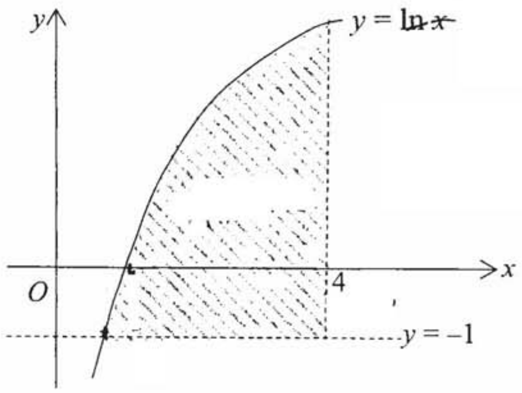
(b) It is given that $y = \dfrac{x}{\sqrt{2x^2 - 1}}$, where $x > 0$. Find the exact value of x when the rate of decrease of y is $\dfrac{9}{8}$ times the rate of increase of x.

9 Given that $y = x \ln x - x$,

(i) show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(x \ln x - x\right) = \ln x$, [2]

(ii) hence show that $\int_a^b (\ln x + 1)\,\mathrm{d}x = b \ln b - a \ln a$. [3]

(iii)



find the area of the shaded region. [3]

4 ln 4 − 3

10 **(a)** Given that $0 < x < \pi$, find the values of x such that

$$\cos\left(\frac{3x}{2}\right) = -\cos\frac{\pi}{10},$$ giving your answers in terms of π. [3]

(b) Prove the identity $\dfrac{1}{1-\sin x}-\dfrac{1}{1+\sin x}=2\tan x\sec x$. [3]

(c) Solve the equation $\sin 4x+3\sin 2x=0$ for $-180^\circ \leq x \leq 180^\circ$. [3]

11 A particle travelling in a straight line passes through a fixed point O with a speed of -10 m/s. The acceleration, a m/s^2, of the particle, t s after passing through O, is given by $a = \frac{24}{(2t+1)^2}$. The particle comes to instantaneous rest at the point P.

(i) Find the time when the particle reaches P. [4]

(ii) Calculate the distance travelled by the particle in the first 3 sec. [3]

(iii) Show that the particle is again at O at some instant during the ninth second after first passing through O. [2]

12 A circle C has a diameter AB where A and B are $(-2, 5)$ and $(12, 11)$ respectively.

(i) Find the equation of the circle C. [3]

The line AB produced intersects another line l which touches the circle C at point $D(8, k)$, where $k > 1$.

(ii) Find the value of k. [1]

(iii) Find the equation of line l. [3]

A chord in the circle C has a midpoint $(12, 8)$.

(iv) Find the coordinates of the points of the intersection of the chord with the circle C. [2]

13 **(a)** Solve the equation $9^x + 8 = 3^{x+2}$. [4]

(b) Without using a calculator, find the value of 20^p given that

$40^{2p-1} = 5^{2-p}$. [3]

(c) Find the value(s) of y that satisfy the equation

$$\log_4(2y) = \log_{16}(y-3) + 3\log_9 3,$$ [4]

2019 Additional Mathematics Prelim Paper 1 Solutions

1 ΔSQR: $\tan 30° = \dfrac{\frac{4}{\sqrt{3}+1}}{QR} \Rightarrow \dfrac{1}{\sqrt{3}} = \dfrac{\frac{4}{\sqrt{3}+1}}{QR}$

$$\Rightarrow QR = \frac{4\sqrt{3}}{\sqrt{3}+1}$$

$$= \frac{4\sqrt{3}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{4\sqrt{3}\left(\sqrt{3}-1\right)}{3-1}$$

$$= 2\left(3-\sqrt{3}\right)$$

$$\therefore\ PR = 2\sqrt{3} + 4\left(3-\sqrt{3}\right)$$

$$= 12 - 2\sqrt{3}$$

$$\therefore \text{Area of trapezium} = \frac{1}{2}\left[2\sqrt{3} + 12 - 2\sqrt{3}\right] \times \frac{4}{\sqrt{3}+1}$$

$$= \frac{24}{\sqrt{3}+1}$$

$$= \frac{24\left(\sqrt{3}-1\right)}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-1\right)}$$

$$= \frac{24\left(\sqrt{3}-1\right)}{3-1}$$

$$= \underline{\underline{12\left(\sqrt{3}-1\right) \text{ units}^2}}$$ [5]

2 $\dfrac{4x^3 + x^2 + 6}{(x-2)\left(x^2+2\right)} = 4 + \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+2}$

Multiplying by $(x-2)\left(x^2+2\right)$, we obtain

$4x^3 + x^2 + 6 = 4(x-2)(x^2+2) + A(x^2+2) + (Bx+C)(x-2)$

Sub $x = 2$: $\quad 4 \times 8 + 4 + 6 = A(4+2)$

$\quad 42 = 6A \Rightarrow A = 7$

Sub $x = 0$: $\quad 6 = -16 + 2(7) + C(-2)$

$\quad -2C = 8 \Rightarrow C = -4$

Compare x^2: $1 = -8 + 7 + B \Rightarrow B = 2$

$\therefore\ \dfrac{4x^3 + x^2 + 6}{(x-2)\left(x^2+2\right)} = \underline{\underline{4 + \dfrac{7}{x-2} + \dfrac{2-4x}{x^2+2}}}$. [5]

3 **(i)** $8x^3 - (x-1)^3 = (2x)^3 - (x-1)^3$

$= \left[2x - (x-1)\right]\left[(2x)^2 + 2x(x-1) + (x-1)^2\right]$

$= (x+1)(4x^2 + 2x^2 - 2x + x^2 - 2x + 1)$

$= (x+1)(7x^2 - 4x + 1)$ [3]

(ii) $8x^3 - (x-1)^3 = 0$

$\Rightarrow (x+1)(7x^2 - 4x + 1) = 0$

$\Rightarrow x = -1$ since for $7x^2 - 4x + 1 = 0$,

$D = (-4)^2 - 4(7) = -12 < 0$

$\therefore\ 7x^2 - 4x + 1 = 0$ has no real roots.

Thus $8x^3 - (x-1)^3 = 0$ has only one real root, -1. [2]

4 **(i)** The minimum point (h, k) lies on the line of symmetry: $x = \dfrac{1+3}{2} = 2$.

$\therefore h = 2$. [1]

(ii) $y = -\left|a(x-h)^2 + k\right|$ where $a > 0$

For (1, 0): $\quad -|a+k| = 0 \Rightarrow a + k = 0$

$\Rightarrow k = -a \quad \ldots (1)$

For (0, –9): $\quad -|4a+k| = -9 \Rightarrow |3a| = 9$

$\Rightarrow 3a = 9$

$a = 3$

$k = -3$

(iii) We find the value of m where the line $y = mx$ is tangent to the curve.

$$\begin{cases} y = mx & \ldots (1) \\ y = 3(x-2)^2 - 3 & \ldots (2) \end{cases}$$

(1) = (2): $\quad mx = 3x^2 - 12x + 9$

$3x^2 - (12+m)x + 9 = 0$

Since the line is tangent to the curve, then $D = 0$,

i.e. $\quad \left[-(12+m)\right]^2 - 4 \times 3 \times 9 = 0$

$(12+m)^2 = 108$

$12 + m = \pm\sqrt{108}$

$= \pm 6\sqrt{3}$

$\therefore\ m = -12 \pm 6\sqrt{3}$

$\therefore\ m = -12 + 6\sqrt{3}$ since $m > -3$

Thus, the line intersects the curve at four distinct points when $\underline{\underline{-12 + 6\sqrt{3} < m < 0}}$. [2]

5 **(i)** gradient of $OQ = \dfrac{2-0}{4-0} = \dfrac{1}{2}$

$\therefore$ gradient of $\perp$ bisector of $OQ = -2$

Midpoint of $OQ = \left(\dfrac{4+0}{2}, \dfrac{2+0}{2}\right)$

$= (2, 1)$

Thus, equation of the perpendicular bisector of OQ is $\quad y - 1 = -2(x-2)$

$\underline{y = -2x + 5}$ [3]

(ii) When $x = 0$, $y = 5$.

When $x = 3$, $y = -1$.

These results show that R and P lie on the perpendicular bisector of OQ.

i.e., RP is the perpendicular bisector of OQ.

Thus, the quadrilateral $OPQR$ is a <u>kite</u>. [2]

(iii) Let $T = (a, b)$

Since $OPQT$ is a rhombus, then midpoint of OQ is the midpoint of RP.

$\therefore\ \left(\dfrac{a+3}{2}, \dfrac{b-1}{2}\right) = (2, 1)$

$\Rightarrow a = 1,\ b = 3$

$\therefore\ \underline{T = (1, 3)}$. [2]

6 $4x^2 + px + q = 0$

(i) Since the roots are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$, then

$\dfrac{1}{\alpha} + \dfrac{1}{\beta} = -\dfrac{p}{4} \Rightarrow \dfrac{\beta + \alpha}{\alpha\beta} = -\dfrac{p}{4}$

$\Rightarrow \dfrac{5}{2} = -\dfrac{p}{4}$

$\therefore p = -10$

$\dfrac{1}{\alpha} \times \dfrac{1}{\beta} = \dfrac{q}{4} \Rightarrow \dfrac{1}{2} = \dfrac{q}{4}$

$\therefore q = 2$

$\underline{p = -10,\ q = 2}$ [4]

(ii) For the new equation, the roots are $\dfrac{2\alpha^2}{\beta}$ and $\dfrac{2\beta^2}{\alpha}$.

$$\text{Sum of roots} = \frac{2\alpha^2}{\beta}+\frac{2\beta^2}{\alpha}$$
$$= \frac{2\left(\alpha^3+\beta^3\right)}{\alpha\beta}$$
$$= \frac{2(\alpha+\beta)\left[\alpha^2-\alpha\beta+\beta^2\right]}{\alpha\beta}$$
$$= \frac{2(\alpha+\beta)\left[(\alpha+\beta)^2-3\alpha\beta\right]}{\alpha\beta}$$
$$= \frac{2(5)\left[5^2-3\times 2\right]}{2}$$
$$= 95$$

$$\text{Product of roots} = \left(\frac{2\alpha^2}{\beta}\right)\left(\frac{2\beta^2}{\alpha}\right)$$
$$= 4\alpha\beta$$
$$= 4\times 2$$
$$= 8$$

Thus the equation is $\underline{x^2-95x+8=0}$. [5]

7 (a) Given: $x^2+ax+2(a-1)>1$

$x^2+ax+(2a-3)>0$

For a positive quadratic function, $D<0$

$\therefore\ a^2-4(2a-3)<0$

$\therefore\ a^2-8a+12<0$

$\therefore\ (a-2)(a-6)<0$

$\therefore\ \underline{2<a<6}$ [3]

(b) $y=3x^2+4x+6$

(i) $y>6 \Rightarrow 3x^2+4x+6>6$

$3x^2+4x>0$

$x(3x+4)>0$

$\therefore\ \underline{\underline{x<-\frac{4}{3} \text{ or } x>0}}$ [2]

(ii) $y=3x^2+4x+6$

$y=-8x-6$

$\therefore\ 3x^2+4x+6=-8x-6$

$3x^2+12x+12=0$

$3(x+2)^2=0$

The equation has equal roots $x=-2$

So the line is tangent to the curve. [2]

8 (a) Given: $y=2x^3-9x^2-1$

gradient: $\dfrac{dy}{dx}=6x^2-18x$

$$\frac{d^2y}{dx^2}=12x-18$$

For minimum gradient, $\dfrac{d^2y}{dx^2}=0$,

i.e., $12x-18=0 \Rightarrow x=\dfrac{3}{2}$

Since $\dfrac{d^3y}{dx^3}=12\ (>0)$, gradient is minimum at $x=\dfrac{3}{2}$.

$$\therefore \text{ Minimum gradient} = 6\left(\frac{3}{2}\right)^2-18\left(\frac{3}{2}\right)$$
$$= \underline{-13.5}$$ [3]

(b) When $y=x^3-6x^2+k$ touches the positive x-axis at A

[This means that the x-axis is tangent to the curve at A and so A is a stationary point.]

(i) $\dfrac{dy}{dx}=3x^2-12x$

At A, $\dfrac{dy}{dx}=0 \Rightarrow 3x(x-4)=0$

$\therefore\ x=0$ or $x=4$

For A, $x>0$, $\therefore\ \underline{A=(4,0)}$ [2]

(ii) A also lies on the curve,

$\therefore\ 0=4^3-6\times 4^2+k$

$\therefore\ \underline{k=32}$ [1]

(iii) $\dfrac{d^2y}{dx^2}=6x-12$

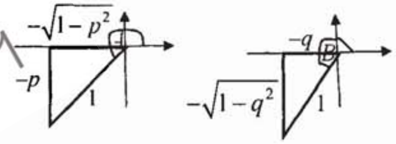
At A, $\dfrac{d^2y}{dx^2}=6(4)-12$

$=12\ (>0)$

Thus A is a minimum point. [2]

9 (a)
$$\frac{5-10\cos^2 A}{\sin A-\cos A}=\frac{5\left(\sin^2 A+\cos^2 A\right)-10\cos^2 A}{\sin A-\cos A}$$
$$=\frac{5\left(\sin^2 A-\cos^2 A\right)}{\sin A-\cos A}$$
$$=\frac{5(\sin A+\cos A)(\sin A-\cos A)}{\sin A-\cos A}$$
$$=5(\sin A+\cos A)$$ [3]

(b) Given: $\sin A=-p$ $\cos B=-q$ and A and B lie in the same quadrant, then they both lie in 3rd quadrant.



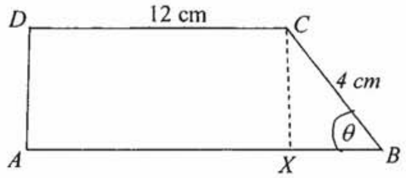
(i) $\sin(-A)=-\sin A$

$=\underline{p}$ [1]

(ii)
$$\tan(45^\circ-A)=\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}$$
$$=\frac{1-\tan A}{1+\tan A} \text{ since } \tan 45^\circ=1$$
$$=\frac{1-\frac{-p}{-\sqrt{1-p^2}}}{1+\frac{-p}{-\sqrt{1-p^2}}}$$
$$=\underline{\underline{\frac{\sqrt{1-p^2}-p}{\sqrt{1-p^2}+p}}}$$ [3]

(iii)
$$\sec 2B=\frac{1}{\cos 2B}$$
$$=\frac{1}{2\cos^2 B-1}$$
$$=\frac{1}{2(-q)^2-1}$$
$$=\underline{\underline{\frac{1}{2q^2-1}}}.$$ [2]

10 **(i)**



$\sin\theta = \dfrac{CX}{4} \;\Rightarrow\; CX = 4\sin\theta$

$\cos\theta = \dfrac{BX}{4} \;\Rightarrow\; BX = 4\cos\theta$

Area of trapezium $= \dfrac{1}{2}[12 + 12 + 4\cos\theta] \times 4\sin\theta$

$= (12 + 2\cos\theta)\, 4\sin\theta$

$= 48\sin\theta + 4(2\sin\theta\cos\theta)$

$= 48\sin\theta + 4\sin 2\theta$ (shown) [3]

(ii) $\dfrac{dA}{d\theta} = 48\cos\theta + 8\cos 2\theta$

For maximum A, $\dfrac{dA}{d\theta} = 0$.

$\therefore 48\cos\theta + 8\cos 2\theta = 0$

$48\cos\theta + 8(2\cos^2\theta - 1) = 0$

$8(2\cos^2\theta + 6\cos\theta - 1) = 0$

$\cos\theta = \dfrac{-6 \pm \sqrt{6^2 - 4\times 2\times(-1)}}{4}$

$\cos\theta = \dfrac{-6 \pm \sqrt{44}}{4}$

$\cos\theta = \dfrac{-6 \pm 2\sqrt{11}}{4}$

$\cos\theta = \dfrac{-3 + \sqrt{11}}{4}$ since θ is acute.

$\therefore \theta = 1.491\ 55$

$\dfrac{d^2A}{d\theta^2} = -48\sin\theta - 16\sin 2\theta$

When $\theta = 1.491\ 55$, $\dfrac{d^2A}{d\theta^2} < 0$,

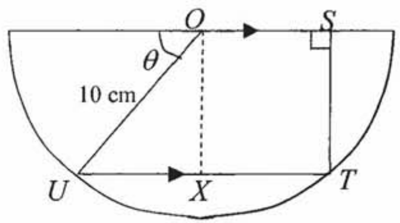
Thus A is maximum when $\underline{\underline{\theta = 1.49}}$. [5]

(iii) Maximum A

$= 48\sin(1.491\ 55) + 4\sin 2(1.491\ 55)$

$= \underline{\underline{53.3 \text{ sq units}}}$ [1]

11 **(i)**



Consider $\Delta\, OUX$,

$\angle OUX = \theta$ (alternate angles)

$\sin\theta = \dfrac{OX}{10} \Rightarrow OX = 10\sin\theta$

$\cos\theta = \dfrac{UX}{10} \;\Rightarrow\; UX = 10\cos\theta$

$L = 10 + UT + ST + OS$

$= 10 + 2\times 10\cos\theta + 10\sin\theta + 10\cos\theta$

$= 10 + 30\cos\theta + 10\sin\theta$ (shown) [3]

(ii) Let $30\cos\theta + 10\sin\theta = R\cos(\theta - \alpha)$

Then $R = \sqrt{30^2 + 10^2} = 10\sqrt{10}$

$\tan\alpha = \dfrac{10}{30} \Rightarrow \alpha = 18.434°$

$\therefore \underline{\underline{L = 10 + 10\sqrt{10}\cos(\theta - 18.4°)}}$ [3]

(iii) Since max $L = 10 + 10\sqrt{10} = 41.6$, it is impossible for the perimeter of $OSTU$ to be 50 cm. [1]

(iv) Given: $L = 35$

$10 + 10\sqrt{10}\cos(\theta - 18.434°) = 35$

$\cos(\theta - 18.434°) = \dfrac{35 - 10}{10\sqrt{10}} = \dfrac{5}{2\sqrt{10}}$

Basic angle = 37.761°

$\therefore \theta - 18.434° = 37.761°$

$\therefore \underline{\underline{\theta = 56.2°}}$ [3]

1 Given: $\frac{d^2y}{dx^2} = 6x - 2$

$\therefore$ Integrating, $\frac{dy}{dx} = 3x^2 - 2x + c$

Given that gradient of normal at $P(2, -8) = -\frac{1}{2}$, this means that gradient of tangent at $P = 2$, i.e. when $x = 2$, $\frac{dy}{dx} = 2$

$\therefore\ 2 = 3(2)^2 - 2(2) + c$

$\Rightarrow c = -6$

$\therefore\ \frac{dy}{dx} = 3x^2 - 2x - 6$

$\therefore$ Integrating, $y = x^3 - x^2 - 6x + c_2$

Since $P(2, -8)$ lies on the curve, i.e. when $x = 2$, $y = -8$,

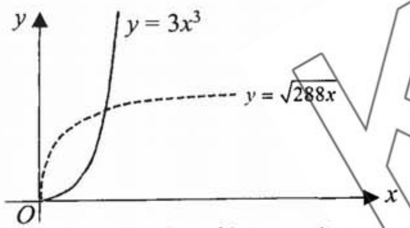
$\therefore\ -8 = 2^3 - 2^2 - 6(2) + c_2$

$\Rightarrow c_2 = 0$

Thus the equation of the curve is

$\underline{\underline{y = x^3 - x^2 - 6x}}$ [6]

2 **(i)**



[2]

(ii) To find the point of intersection, we solve the equations simultaneously:

$y = 3x^3$ ... (1)

$y = \sqrt{288x}$ ... (2)

(1) = (2): $3x^3 = \sqrt{288x}$

$9x^6 = 288x$

$9x^6 - 288x = 0$ **[Do not divide by variable x]**

$9x(x^5 - 32) = 0$

$\therefore x = 0$ or $x = 2$

$y = 0$ or $x = 24$

So the 2 points of intersection are (0, 0) and (2, 24).

Gradient of line joining these 2 points $= \frac{24 - 0}{2 - 0}$

$= 12$

$y = 3x^3$

$\frac{dy}{dx} = 9x^2$

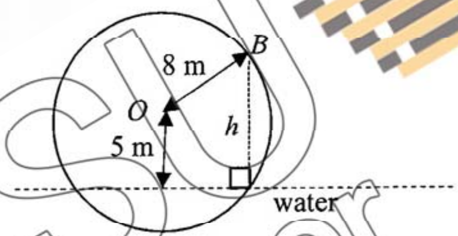
Since gradient of tangent to the curve at A is parallel to the line passing through the 2 points of intersection,

$\therefore 9x^2 = 12$

$x^2 = \frac{4}{3}$

$\therefore\ x = \frac{2}{\sqrt{3}} = \underline{\underline{\frac{2\sqrt{3}}{3}}}$ since $x \geq 0$. [4]

3



Given: $h = a\cos bt + c$

(i) Starting point is when B is at its highest point, i.e., when $t = 0$, $h = 13$

$\therefore 13 = a(1) + c \Rightarrow a + c = 13$ ... (1)

and lowest point is when B is 3 m below water level.

$\therefore -3 = a(-1) + c \Rightarrow -a + c = -3$ ... (2)

Solving (1) and (2),

$\underline{\underline{a = 8, c = 5}}$

Given: 5 revolutions take 1 minute

$\therefore$ 1 revolution takes $\frac{1}{5}$ minute (= 12 seconds)

So, period: $\frac{2\pi}{b} = 12 \Rightarrow \underline{\underline{b = \frac{\pi}{6}}}$ [3]

$\therefore\ h = 8\cos\left(\frac{\pi}{6}t\right) + 5$

(ii) $h < 0 \Rightarrow 8\cos\left(\frac{\pi}{6}t\right) + 5 < 0$

When $h = 0$, $8\cos\left(\frac{\pi}{6}t\right) + 5 = 0$

$\cos\left(\frac{\pi}{6}t\right) = -\frac{5}{8}$

Basic angle $= \cos^{-1}\left(\frac{5}{8}\right) = 0.895\ 66$

The variable angle $\frac{\pi}{6}t$ lies in the 2nd and 3rd quadrants,

$\therefore \frac{\pi}{6}t = \pi - 0.895\ 66$ or $\pi + 0.895\ 66$ in the 1st revolution

$t = 4.289$ or 7.710

so duration $= 7.710 - 4.289$

$= \underline{3.42 \text{ seconds}}$ [3]

4 **(i)** For $x\left(2x + \frac{k}{x}\right)^8$, $T_{r+1} = x\binom{8}{r}(2x)^{8-r}\left(\frac{k}{x}\right)^r$

Power of $x = 1 + (8 - r) + (-r)$

$= 9 - 2r$

For term in x^3, power of $x = 3$

$\therefore 9 - 2r = 3$

$\Rightarrow r = 3$

$\therefore$ Term in $x^3 = x\binom{8}{3}(2x)^5\left(\frac{k}{x}\right)^3$

$= \ldots$

$= 1\ 792\ k^3$

Since coefficient of $x^3 = 28$,

$1\ 792\ k^3 = 28$

$\Rightarrow k = \frac{1}{4}$ (shown) [4]

(ii) For constant term, power of $x = 0$,

$\therefore 9 - 2r = 0$

$\Rightarrow r = \frac{9}{2}$

Since r is not a <u>whole number</u> (or <u>positive integer</u>), then we can conclude that there is no constant term. [1]

(iii) In the expansion of $x\left(2x + \frac{k}{x}\right)^8 + k(1 - x)^{10}$,

Term in $x^3 = 28x^3 + \frac{1}{4} \times \binom{10}{3}(-x)^3$

$= 28x^3 + \frac{1}{4} \times (-120x^3)$

$= -2x^3$

$\therefore$ <u>coefficient of $x^3 = -2$</u> [2]

5 **(i)** <u>To show</u>: AB is parallel to DC

This means that we need only to show that there are 2 equal angles that satisfy either alternate angles, corresponding angles or interior angles
Here we can show $\angle CDT = \angle BAT$]

$\angle CDT = \angle DTP$ (alternate segment theorem)
$= \angle ATP$ (common angle)
$= \angle BAT$ (alternate segment theorem)

Since this result satisfies the properties of corresponding angles, then $AB \,//\, DC$. (shown) [2]

(ii) <u>To show</u>: the line TE bisects $\angle ATB$

[This means that we need to show: $\angle ATE = \angle ETB$]

$\angle ATE = \angle DTE$ (common angle)
$= \angle DCE$ (angle in the same segment)
$= \angle CEB$ (alternate angles since $AB//DC$)
$= \angle ETB$ (alternate segment theorem)

Thus, TE bisects $\angle ATB$. (show) [3]

(iii) <u>To show</u>: $\Delta DFT \equiv \Delta EFC$ if $DF = EF$

From (ii), $\angle DTE = \angle DCE$ (angle in same segment)
i.e. $\angle DTF = \angle ECF$ (common angles) A
$DF = EF$ (given) S
and $\angle DFT = \angle EFC$ (vertically opp angles) A

Thus, $\Delta DFT \equiv \Delta EFC$ (AAS) [2]

6 Given: $\underbrace{y-x}_{Y} = \frac{b}{a}x^2 + b$ **[note that this is already in linear form]**

$Y \;=\; m\,X \;+\; c$

(i) Plotting $y - x$ against x^2 will give a straight line.

x^2	1	4	9	16
$y-x$	1.73	5.5	11.75	20.5

(ii) $b = Y-$ intercept
$= 0.5$

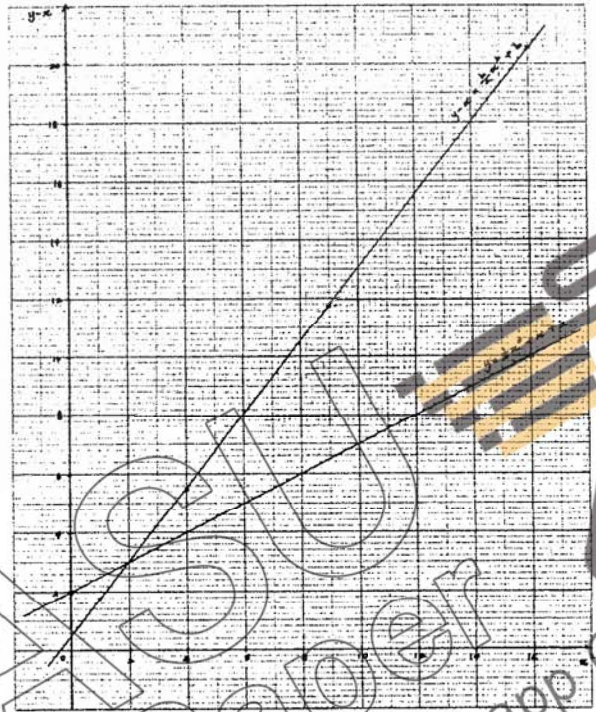
$\frac{b}{a}$ = gradient of line

$= \frac{20.5-0.5}{16-0}$

$= 1.25$

$\therefore \frac{0.5}{a} = 1.25 \Rightarrow a = 0.4$

$\therefore$ <u>$a = 0.4, b = 0.5$</u> [2]



[3]

(iii) $y = \frac{1}{2}x^2 + x + 2 \Rightarrow y - x = \frac{1}{2}x^2 + 2$

Using the same axes, $Y = y - x$ and $X = x^2$,

Then we need to draw the line $Y = \frac{1}{2}X + 2$.

When $X = 0$, $Y = 2$ and when $X = 16$, $Y = 10$

So the line passes through (0, 2) and (16, 10)

The point of intersection between the 2 lines is approximately (2, 3).

i.e. $X = 2 \Rightarrow x^2 = 2$, $\quad \therefore x = 1.41$ (3 sf)

$Y = 3 \Rightarrow y - x = 3$, $\therefore y = 3 + 1.41 = 4.41$

<u>$x = 1.41$ and $y = 4.41$</u> [2]

7 **(i)** Since $(2x^2 + x - 1)$ is a factor of P(x), then we can write P(x) = $(2x^2 + x - 1)\,Q(x)$. However, we know that $Q(x)$ is a polynomial of degree 2.

So, P(x) = $(2x^2 + x - 1)(ax^2 + bx + c)$.

However, we know that for P(x), coeff of x^2 = and constant term = –3.

This tells us that $a = 1$ and $c = 3$.

So, P(x) = $(2x^2 + x - 1)(x^2 + bx + 3)$.

[Writing P(x) in this form, we only need to find one unknown, i.e. b.]

Given: when divided by $(x - 1)$, P(x) has remainder 4, i.e. P(1) = 4.

$\therefore 4 = (2 + 1 - 1)(1 + b + 3)$
$= 2(b + 4)$
$b + 4 = 2 \Rightarrow b = -2.$
$\therefore$ P(x) = $(2x^2 + x - 1)(x^2 - 2x + 3)$
$=$ <u>$(2x - 1)(x + 1)(x^2 - 2x + 3)$</u> [4]

(ii) P(x) = 0

$\Rightarrow$ <u>$x = \frac{1}{2}$ or -1</u> since $x^2 - 2x + 3 = 0$ has no real roots as D = $(-2)^2 - 4\times3 = -8 < 0$. [1]

(iii) P($1 - x$) = 0

$\Rightarrow 1 - x = \frac{1}{2}$ or -1

$\therefore$ <u>$x = \frac{1}{2}$ or 2</u> [2]

8 **(a)** $y = x^2e^{3x}$

$\therefore \frac{dy}{dx} = 2xe^{3x} + 3x^2e^{3x}$
$= xe^{3x}(2 + 3x)$

For increasing function, $\frac{dy}{dx} > 0$

$\therefore xe^{3x}(2 + 3x) > 0$

Since $e^{3x} > 0$ for all real values of x.

$\therefore x(2 + 3x) > 0$

$\therefore$ <u>$x < -\frac{2}{3}$ or $x > 0$</u> [4]

(b) $y = \dfrac{x}{\sqrt{2x^2-1}}$ where $x > 0$

$$\frac{dy}{dx} = \frac{\sqrt{2x^2-1} - x \times \frac{1}{2} \times \frac{4x}{\sqrt{2x^2-1}}}{2x^2-1}$$

$$= \frac{\sqrt{2x^2-1} - x \times \frac{1}{2} \times \frac{4x}{\sqrt{2x^2-1}}}{2x^2-1} \times \frac{\sqrt{2x^2-1}}{\sqrt{2x^2-1}}$$

$$= \frac{(2x^2-1) - 2x^2}{\sqrt{(2x^2-1)^3}}$$

$$= -\frac{1}{\sqrt{(2x^2-1)^3}}$$

Given: $\dfrac{dy}{dt} = -\dfrac{9}{8}\dfrac{dx}{dt}$ $\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{9}{8}$

$$\therefore -\frac{1}{\sqrt{(2x^2-1)^3}} = -\frac{9}{8}$$

$$\therefore \sqrt{(2x^2-1)^3} = \frac{8}{9}$$

$$(2x^2-1)^3 = \frac{64}{81}$$

$$2x^2 - 1 = \sqrt[3]{\frac{64}{81}}$$

$$x^2 = \frac{1}{2}\left(1 + \frac{4}{\sqrt[3]{81}}\right)$$

$$\therefore x = \sqrt{\frac{1}{2} + \frac{2}{\sqrt[3]{81}}} \quad \text{since } x > 0$$

$$= 0.981 \text{ (3 sf)}$$

[4]

9 Given: $y = x\ln x - x$

(i) $\dfrac{dy}{dx} = \left(\ln x + x \times \dfrac{1}{x}\right) - 1$

$= (\ln x + 1) - 1$

$= \ln x$ (shown) [2]

(ii) From (i), we know that $\int \ln x \, dx = x\ln x - x + c$

$$\int_a^b (\ln x + 1)\,dx = \int_a^b \ln x \, dx + \int_a^b 1 \, dx$$

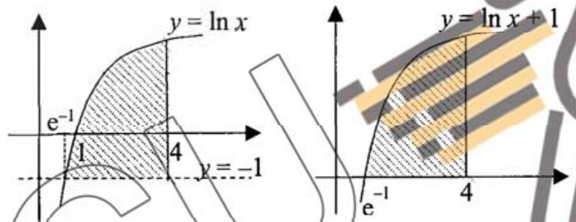
$$= [x\ln x - x]_a^b + [x]_a^b$$

$$= (b\ln b - b) - (a\ln a - a) + (b - a)$$

$$= b\ln b - a\ln a \text{ (shown)}$$

[3]

(iii) The shaded area is the same in both diagrams:



When $\ln x = -1$, $x = e^{-1} = \dfrac{1}{e}$.

Shaded area $= \displaystyle\int_{\frac{1}{e}}^{4} (\ln x + 1)\,dx$

$= 4\ln 4 - \dfrac{1}{e}\ln\dfrac{1}{e}$ [using (ii)]

$= \underline{5.91 \text{ units}^2}$ [3]

10 (a) $\cos\left(\dfrac{3x}{2}\right) = -\cos\dfrac{\pi}{10}$ for $0 < x < \pi$

basic angle $= \dfrac{\pi}{10}$

The variable angle $\dfrac{3x}{2}$ lies in 2nd and 3rd quadrants.

$\therefore \dfrac{3x}{2} = \pi - \dfrac{\pi}{10}$ or $\pi + \dfrac{\pi}{10}$

$= \dfrac{9\pi}{10}$ or $\dfrac{11\pi}{10}$

$\therefore x = \dfrac{3\pi}{5}$ or $\dfrac{11\pi}{15}$ [3]

(b) $\dfrac{1}{1-\sin x} - \dfrac{1}{1+\sin x} = \dfrac{(1+\sin x) - (1-\sin x)}{(1-\sin x)(1+\sin x)}$

$= \dfrac{2\sin x}{1-\sin^2 x}$

$= \dfrac{2\sin x}{\cos^2 x}$

$= 2\dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos x}$

$= 2\tan x \sec x$ (shown) [3]

(c) $\sin 4x + 3\sin 2x = 0$ for $-180° \le x \le 180°$

$2\sin 2x\cos 2x + 3\sin 2x = 0$

$\sin 2x(2\cos 2x + 3) = 0$

$\therefore \sin 2x = 0$ or $\cos 2x = -\dfrac{3}{2}$ (no solution)

basic angle $= 0°$

$\therefore 2x = -360°, -180°, 0°, 180°$ or $360°$

$x = -180°, -90°, 0°, 90°$ or $180°$ [3]

11 Given: $a = \dfrac{24}{(2t+1)^2}$

(i) $v = \dfrac{24(2t+1)^{-1}}{2(-1)} + c$

$= -\dfrac{12}{2t+1} + c$

when $t = 0$, $v = -10$ m/s

$\therefore c = 2$

$\therefore v = 2 - \dfrac{12}{2t+1}$

At P, $v = 0 \Rightarrow 2 - \dfrac{12}{2t+1} = 0$

$\Rightarrow \underline{t = 2.5 \text{ s}}$ [4]

(ii) $s = 2t - 12\dfrac{\ln(2t+1)}{2} + c_1$

$= 2t - 6\ln(2t+1) + c_1$

when $t = 0$, $s = 0$, $\therefore c_1 = 0$

$\therefore s = 2t - 6\ln(2t+1)$

$t = 0$, $s = 0$

$t = 2.5$, $s = 2(2.5) - 6\ln 6 = -5.750\,5$

$t = 3$, $s = 2(3) - 6\ln 7 = -5.675\,4$

Distance travelled $= 5.750\,5 + (5.750\,5 - 5.675\,4)$

$= \underline{5.83 \text{ m}}$ (3 sf) [3]

(iii) **[9th second means from $t = 8$ s to $t = 9$ s]**

When $t = 8$, $s = 2(8) - 6\ln 17 = -0.999\ 28$ m

When $t = 9$, $s = 2(9) - 6\ln 19 = +0.333\ 36$ m

$\therefore s = 0$ for $8 < t < 9$

i.e. The particle is again at O during the 9th sec. [2]

12 **(i)** centre, X = midpoint of AB

$$= \left(\frac{-2+12}{2}, \frac{5+11}{2}\right)$$

$$= (5, 8)$$

diameter = AB

$$= \sqrt{(12+2)^2 + (11-5)^2}$$

$$= \sqrt{232}$$

$\therefore$ Equation of circle is

$$(x-5)^2 + (y-8)^2 = \left(\frac{\sqrt{232}}{2}\right)^2$$

$$\underline{\underline{(x-5)^2 + (y-8)^2 = 58}}$$ [3]

(ii) Since $D(8, k)$ lies on the circle, then

$(8-5)^2 + (k-8)^2 = 58$

$(k-8)^2 = 49$

$\therefore k - 8 = 7$ or -7

$k = 15$ or 1

Since $k > 1$, $\therefore \underline{\underline{k = 15}}$. [1]

(iii) gradient of $DX = \dfrac{15-8}{8-5} = \dfrac{7}{3}$

Since line l is perpendicular to radius DX,

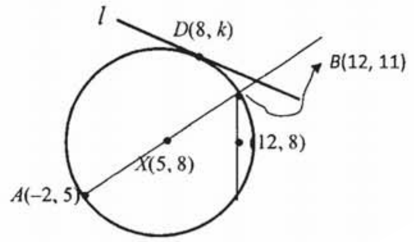
$\therefore$ gradient of $l = -\dfrac{3}{7}$

So, equation of line l is

$$y - 15 = -\frac{3}{7}(x-8)$$

$$y = -\frac{3}{7}x + \frac{24}{7} + 15$$

$$\underline{\underline{y = -\frac{3}{7}x + \frac{129}{7}}}$$ [3]



(iv) Given that (12, 8) is the midpoint of the chord, and since $B(12, 11)$ is vertically above (12, 8), then the chord is a vertical line with $x = 12$.

The other end-point of the chord is (12, 5)

Thus the points of intersection are

$\underline{\underline{(12, 5) \text{ and } (12, 11)}}$. [2]

13 **(a)** Given: $9^x + 8 = 3^{x+2}$

$$3^{2x} + 8 = 3^x \times 3^2$$

Let $u = 3^x$, then the equation becomes

$u^2 + 8 = 9u$

$u^2 - 9u + 8 = 0$

$(u-1)(u-8) = 0$

$u = 1$ or $u = 8$

$3^x = 1$ or $3^x = 8$

$\therefore x = 0$ or $x = \log_3 8$

$$= \frac{\lg 8}{\lg 3}$$

$$= 1.89 \text{ (3 sf)}$$ [4]

(b) $40^{2p-1} = 5^{2-p} \Rightarrow \dfrac{40^{2p}}{40} = \dfrac{5^2}{5^p}$

$\therefore 40^{2p} \times 5^p = 5^2 \times 40$

$(40^2 \times 5)^p = 1000$

$(8000)^p = 1000$

$(20)^{3p} = 10^3$

$\therefore \underline{\underline{20^p = 10}}$ [3]

(c) $\log_4(2y) = \log_{16}(y-3) + 3\log_9 3$

$$\log_4(2y) = \frac{\log_4(y-3)}{\log_4 16} + 3\frac{\log_3 3}{\log_3 9}$$

$$\log_4(2y) = \frac{\log_4(y-3)}{2} + \frac{3}{2}$$

$$2\log_4(2y) = \log_4(y-3) + 3$$

$$\log_4(2y)^2 - \log_4(y-3) = 3$$

$$\log_4\frac{(2y)^2}{y-3} = 3$$

$$\therefore \frac{4y^2}{y-3} = 4^3$$

$$4y^2 = 64(y-3)$$

$$y^2 = 16(y-3)$$

$$y^2 - 16y + 48 = 0$$

$$(y-4)(y-12) = 0$$

$\therefore \underline{\underline{y = 4 \text{ or } y = 12}}$. [4]