ANDERSON SERAGOON JUNIOR COLLEGE
JC2 Preliminary Examinations 2019
Higher 3

H3 MATHEMATICS 9820/01

Paper 1

20 September 2019

3 hours

Additional Materials: Answer Booklet

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class, syllabus and paper number on the answer booklet you hand in.
Write in dark blue or black pen on both sides of the answer booklet.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, ensure that you have submitted all your work.
The number of marks is given in brackets [ ] at the end of each question or part question.

1 **(a)** Show that $\gcd\left(a+b, a^2-ab+b^2\right)=\gcd\left(a+b, 3ab\right)$ for non-zero integers a and b.
If $\gcd\left(a,b\right)=1$, show that $\gcd\left(a+b, a^2-ab+b^2\right)$ is either 1 or 3. [7]

(b) Show that $\gcd\left(n!+1, (n+1)!+1\right)=1$, where n is a positive integer. [4]

2 **(i)** It is given that all the terms in the two sequences $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ are non-zero, where $n \geq 1$. By considering the function $\mathrm{f}(x)=\sum_{r=1}^{n}(p_r x - q_r)^2$, prove that

$$\left(\sum_{r=1}^{n} p_r^{\ 2}\right)\left(\sum_{r=1}^{n} q_r^{\ 2}\right) \geq \left(\sum_{r=1}^{n} p_r \cdot q_r\right)^2,$$

and that equality holds if and only if $\frac{q_1}{p_1}=\frac{q_2}{p_2}=\cdots=\frac{q_n}{p_n}$. [4]

(ii) Use the result in **(i)** to show that if a, b, c and d are positive real numbers, then
$$\left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right) \geq abcd(a+b+c+d)^2. \qquad [3]$$

(iii) Hence show that $\left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right) \geq 16\left(abcd\right)^{\frac{3}{2}}$.
Find a condition relating a, b, c and d for equality to hold, explaining your answer. [4]

3 **(a)** Find $\displaystyle\int \frac{\cos x}{\sin x - \cos^2 x}\,\mathrm{d}x$. [4]

(b) **(i)** For a twice differentiable function f, if $\mathrm{f}(x)$ satisfies the differential equation $\mathrm{f}''(x)=\lambda \mathrm{f}'(x)\mathrm{f}(x)$ for some real constant λ, express $\displaystyle\int [\mathrm{f}'(x)]^2 \cdot [\mathrm{f}(x)]^N \,\mathrm{d}x$ in terms of $\mathrm{f}(x)$, $\mathrm{f}'(x)$, λ and N, where N is a positive integer. [4]

(ii) Verify that the above result can be applied when $\mathrm{f}(x)=\tan 4x$.
Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{16}} \frac{\sin^{246} 4x}{\cos^{250} 4x}\,\mathrm{d}x$. [4]

4 For two integers x and y, let their highest common factor be denoted by $\text{hcf}(x, y)$ and their lowest common multiple by $\text{lcm}(x, y)$.

Given that a and b are positive integers,

(i) show that there exists integers x and y such that $\text{hcf}(x, y) = a$ and $\text{lcm}(x, y) = b$ if and only if a is a divisor of b. [4]

(ii) if a is a divisor of b and r is the number of distinct primes in the prime factorisation of $\frac{b}{a}$, show that the number of distinct ordered pairs of positive integers, (x, y), such that $\text{hcf}(x, y) = a$ and $\text{lcm}(x, y) = b$, is 2^r. [5]

5 Happie Newz Café sells 3 types of muffins, namely chocolate, strawberry and vanilla. The café owner bakes a large number of muffins of each type. All muffins of the same flavour are taken to be identical to one another.

(i) The café owner wants to put 20 muffins in a row on a glass panel so as to entice customers to buy muffins. Find the number of ways she can do so if

(a) there are no restrictions, [1]

(b) no 2 muffins of the same type are adjacent. [2]

(ii) Mr Tay wants some muffins for his 6 students. Find the number of ways that he can do so if

(a) every student gets at least 1 muffin and no student gets 2 or more muffins of the same type, [2]

(b) every student gets exactly 4 muffins and no 2 students get the same combination of muffins. [3]

(iii) At the end of the day, the café owner is left with 6 chocolate, 7 strawberry and 8 vanilla muffins. She wants to throw away these muffins into the 4 different rubbish bins in the vicinity of the café. She wants to ensure that at least 1 muffin is thrown into each bin so as to avoid detection by the members of public that she is wasting food. Find the number of ways she can do so. [5]

6 **(i)** Let $L_{n,k}$ denote the number of ways to distribute *n* distinct objects into *k* identical boxes, such that no box is empty and the ordering of objects within each box matters.

(a) Find $L_{n,n}$ and $L_{n,1}$, giving your answer in terms of *n* whenever possible. [2]

(b) Explain why $L_{n,n-1} = n^2 - n$ for $n \geq 2$. [2]

(c) Show that $L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$ for $n \geq k \geq 2$. [3]

(ii) Let $l_{n,k}$ denote the number of ways to distribute *n* distinct objects into *k* distinct boxes, such that no box is empty and the ordering of objects within each box matters.

(a) Use part **(i)(c)** to show that $l_{n,k} = k\,l_{n-1,k-1} + (n+k-1)l_{n-1,k}$ for $n \geq k \geq 2$. [2]

(b) Explain why $\dfrac{l_{n,k}}{n!} = \dbinom{n-1}{k-1}$. [2]

(c) Hence, by using bijective principle, or otherwise, show that for $n \geq k$,

$$\sum_{j=1}^{k} \binom{k}{j}\binom{n-1}{j-1} = \frac{(n+k-1)!}{n!(k-1)!}$$ [5]

7 Two sequences $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are given by

$$u_1 = 1, \quad v_1 = 1, \quad \text{and}$$
$$u_{n+1} = u_n + 3v_n, \quad v_{n+1} = 2u_n + 7v_n,$$

for positive integers *n*. The sequence $r_1, r_2, r_3, \ldots$ is such that $r_n = \dfrac{u_n}{v_n}$ for positive integers *n*.

(i) Use induction to prove that $2u_n{}^2 - 3v_n{}^2 + 6u_n v_n = 5$ for all positive integers *n*. [4]

It is given that as $n \to \infty$, $v_n \to \infty$ and $r_n \to L$ for some real constant *L*.

(ii) Using the results in **(i)** or otherwise, show that $L = \frac{1}{2}\left(-3+\sqrt{15}\right)$. [3]

(iii) Describe the behaviour of the sequence $r_1, r_2, r_3, \ldots$, justifying your answer by considering $r_{n+1} - r_n$. Hence, show that $\sqrt{15}$ lies within the interval $\left(\dfrac{15}{2r_n+3},\ 2r_n+3\right)$ for all positive integers *n*. [4]

(iv) Deduce that $\dfrac{213}{55} < \sqrt{15} < \dfrac{275}{71}$. [2]

8 For any positive integer x, $\phi(x)$ is defined to be the number of positive integers not exceeding x which are coprime to x. For example, $\phi(4)=2$ since positive integers not exceeding 4 and coprime to 4 are 1 and 3.

(i) Find $\phi(p)$ when p is prime. [1]

(ii) Show that, if p is prime and r is a positive integer, then $\phi(p^r)=p^{r-1}(p-1)$. [2]

It is given that $\phi(mn)=\phi(m)\phi(n)$ for positive integers m and n such that $\gcd(m,n)=1$.

(iii) Show that $\phi(x)$ is even for all $x \geq 3$. [5]

(iv) Find all positive integers x such that $\phi(3x)=\phi(2x)$. [7]

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1	**(a)** Let $\gcd\left(a+b, a^2-ab+b^2\right)=k$. Then $k\mid(a+b)$ and $k\mid\left(a^2-ab+b^2\right)$ ie $k\mid(a+b)^2 \Rightarrow k\mid\left(a^2+2ab+b^2\right)$ $\Rightarrow k\mid\left(\left(a^2+2ab+b^2\right)-\left(a^2-ab+b^2\right)\right)$ $\Rightarrow k\mid 3ab$ $\Rightarrow \gcd\left(a+b, a^2-ab+b^2\right)\le \gcd(a+b, 3ab)$ Let $\gcd(a+b, 3ab)=h$ Then $h\mid(a+b)$ and $h\mid 3ab$ ie $h\mid(a+b)^2 \Rightarrow h\mid\left(a^2+2ab+b^2\right)$ $\Rightarrow h\mid\left(\left(a^2+2ab+b^2\right)-3ab\right)$ $\Rightarrow h\mid\left(a^2-ab+b^2\right)$ $\Rightarrow \gcd(a+b, 3ab)\le \gcd\left(a+b, a^2-ab+b^2\right)$ Hence $\gcd\left(a+b, a^2-ab+b^2\right)=\gcd(a+b, 3ab)$. Let $s=\gcd\left(a+b, a^2-ab+b^2\right)=\gcd(a+b, 3ab)$ Then $s\mid(a+b)$ and $s\mid 3ab$ If $s\mid a$, then $s\mid((a+b)-a)\Rightarrow s\mid b\Rightarrow s=1$ since $\gcd(a,b)=1$. Similarly, if $s\mid b$, then $s\mid((a+b)-b)\Rightarrow s\mid a\Rightarrow s=1$ since $\gcd(a,b)=1$. If $s\nmid a$ and $s\nmid b$, then from $s\mid 3ab$ and $\gcd(a,b)=1$, we have $s\mid 3$. Therefore s = 1 or 3 ie $\gcd\left(a+b, a^2-ab+b^2\right)=1$ or 3. **(b)** Let $\gcd(n!+1, (n+1)!+1)=k$ $k\mid(n!+1)\Rightarrow n!+1=sk$ for some $s\in Z$ $k\mid((n+1)!+1)\Rightarrow (n+1)!+1=tk$ for some $t\in Z$ $\Rightarrow (n!)(n+1)+1=tk$ $\Rightarrow (sk-1)(n+1)+1=tk$ $\Rightarrow skn+sk-n-1+1=tk$ $\Rightarrow n=k(sn+s-t)$ $\Rightarrow k\mid n$ $\Rightarrow k\mid n!$ $\Rightarrow k\mid((n!+1)-n!)$ $\Rightarrow k\mid 1$ Therefore k = 1 ie $\gcd(n!+1, (n+1)!+1)=1$.

2	**(i)** $\mathrm{f}(x)=\sum_{r=1}^{n}(p_r x-q_r)^2$ $=\sum_{r=1}^{n}(p_r{}^2x^2-2p_rq_rx+q_r{}^2)$ $=\left(\sum_{r=1}^{n}p_r{}^2\right)x^2-\left(2\sum_{r=1}^{n}p_rq_r\right)x+\left(\sum_{r=1}^{n}q_r{}^2\right)$ Since $(p_rx-q_r)^2\ge 0$, $\mathrm{f}(x)\ge 0$ for all x. Therefore, discriminant ≤ 0. $\left(2\sum_{r=1}^{n}p_rq_r\right)^2-4\left(\sum_{r=1}^{n}p_r{}^2\right)\left(\sum_{r=1}^{n}q_r{}^2\right)\le 0$ $\left(\sum_{r=1}^{n}p_r{}^2\right)\left(\sum_{r=1}^{n}q_r{}^2\right)\ge\left(\sum_{r=1}^{n}p_rq_r\right)^2$ Equality holds if and only if the graph of $y=\mathrm{f}(x)$ touches the x-axis, i.e. $p_rx-q_r=0$ for $r=1,2,\ldots,n \quad \Leftrightarrow x=\frac{q_1}{p_1}=\frac{q_2}{p_2}=\cdots=\frac{q_n}{p_n}$. **(ii)** Using part **(i)** result with n = 4, $\left[\left(\frac{a}{\sqrt{cd}}\right)^2+\left(\frac{b}{\sqrt{ad}}\right)^2+\left(\frac{c}{\sqrt{ab}}\right)^2+\left(\frac{d}{\sqrt{bc}}\right)^2\right]\left[\left(\sqrt{cd}\right)^2+\left(\sqrt{ad}\right)^2+\left(\sqrt{ab}\right)^2+\left(\sqrt{bc}\right)^2\right]\ge(a+b+c+d)^2$ $\Leftrightarrow \left(\frac{d^2}{bc}+\frac{a^2}{cd}+\frac{b^2}{ad}+\frac{c^2}{ab}\right)\left(ab+bc+cd+da\right)\ge(a+b+c+d)^2$ $\Leftrightarrow \left(ad^3+ba^3+cb^3+dc^3\right)\left(ab+bc+cd+da\right)\ge abcd(a+b+c+d)^2$ **(iii)** By AM-GM inequality, $a+b+c+d\ge 4\cdot\left(abcd\right)^{\frac{1}{4}}$. Therefore, $abcd(a+b+c+d)^2\ge abcd\cdot 16(abcd)^{1/2}=16\left(abcd\right)^{3/2}$, and the result follows. For equality to hold for the AM-GM inequality, $a=b=c=d$. For equality to hold for the Cauchy-Schwarz inequality, $\frac{cd}{a}=\frac{ad}{b}=\frac{ab}{c}=\frac{bc}{d}$, which will be satisfied when $a=b=c=d$.
3	**(a)** $\int\frac{\cos x}{\sin x-\cos^2 x}\,\mathrm{d}x=\int\frac{\cos x}{\sin^2 x+\sin x-1}\,\mathrm{d}x$ $=\int\frac{\cos x}{\left(\sin x+\frac{1}{2}\right)^2-\frac{5}{4}}\,\mathrm{d}x$ $=\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\ln\left\lvert\frac{\sin x+\frac{1}{2}-\frac{\sqrt{5}}{2}}{\sin x+\frac{1}{2}+\frac{\sqrt{5}}{2}}\right\rvert+c=\frac{1}{\sqrt{5}}\ln\frac{\lvert 2\sin x+1-\sqrt{5}\rvert}{2\sin x+1+\sqrt{5}}+c$

	(b) $\int [\mathrm{f}'(x)]^2[\mathrm{f}(x)]^N \,\mathrm{d}x = \int \mathrm{f}'(x)\cdot \mathrm{f}'(x)[\mathrm{f}(x)]^N \,\mathrm{d}x$ $= \frac{1}{N+1}[\mathrm{f}(x)]^{N+1}\mathrm{f}'(x) - \frac{1}{N+1}\int\left([\mathrm{f}(x)]^{N+1}\mathrm{f}''(x)\right)\mathrm{d}x$ [integrate by parts] $= \frac{1}{N+1}[\mathrm{f}(x)]^{N+1}\mathrm{f}'(x) - \frac{1}{N+1}\int\left(\lambda \mathrm{f}'(x)[\mathrm{f}(x)]^{N+2}\right)\mathrm{d}x$ $= \frac{1}{N+1}[\mathrm{f}(x)]^{N+1}\mathrm{f}'(x) - \frac{\lambda}{(N+1)(N+3)}[\mathrm{f}(x)]^{N+3} + c$ $\mathrm{f}(x) = \tan 4x,\ \mathrm{f}'(x) = 4\sec^2 4x,\ \mathrm{f}''(x) = 32\sec^2 4x \tan 4x = 8\mathrm{f}'(x)\mathrm{f}(x).$ So $\lambda = 8.$ $\int_0^{\frac{\pi}{16}} \frac{\sin^{246} 4x}{\cos^{250} 4x}\,\mathrm{d}x = \frac{1}{16}\int_0^{\frac{\pi}{16}} [4\sec^2 4x]^2(\tan 4x)^{246}\,\mathrm{d}x$ $= \frac{1}{16}\left[\frac{1}{247}\tan^{247} 4x \cdot 4\sec^2 4x - \frac{8\tan^{249} 4x}{247\times 249}\right]_0^{\frac{\pi}{16}}$ $= \frac{1}{16}\left[\frac{1}{247}(1)(4)(2) - \frac{8}{247\times 249}\right]$ $= \frac{1}{2}\left(\frac{249-1}{247\times 249}\right) = \frac{124}{61503}$
4	$(\Leftarrow)$ If $a\mid b \Rightarrow b = ak$ for some $k\in Z$. Let $x = a$ and $y = b = ak$ Then $hcf\,(x,y) = (a,ak) = a$ and $lcm(x,y) = lcm(a,ak) = ak = b$. $(\Rightarrow)$ There exists integers x, y such that $hcf\,(x,y) = a$ and $lcm(x,y) = b$. Then $x = as$ and $y = at$ where $\gcd(s,t) = 1$ $lcm(x,y) = lcm(as,at) = b \Rightarrow b = ast$ Therefore a is a divisor of b. Let $\frac{b}{a} = p_1^{k_1}p_2^{k_2}.....p_r^{k_r}$ where $p_1, p_2, ...p_r$ are the r distinct prime factors of $\frac{b}{a}$ and $k_1, k_2, ..., k_r$ are non-negative integers. Then $b = ap_1^{k_1}p_2^{k_2}.....p_r^{k_r}$ $hcf\,(x,y) = a$ and $lcm(x,y) = b \Rightarrow x = as,\quad y = at,\quad b = ast$ where $\gcd(s,t) = 1$. $ast = ap_1^{k_1}p_2^{k_2}.....p_r^{k_r}$ $st = p_1^{k_1}p_2^{k_2}.....p_r^{k_r}$ Since $\gcd(s,t) = 1$, then s is the product of some prime factors each of the form $p_i^{k_1}$ where t is the product of the remaining prime factors of the form $p_j^{k_j}$ Each $p_i^{k_1}$ may or may not be a factor of s which means that there are two choices for each $p_i^{k_1}$ with regard to s. Thus the total number of possible integers s is 2^r with t taking the leftovers. Hence the number of distinct ordered pairs of positive integers (x,y) such that $hcf\,(x,y) = a$ and $lcm(x,y) = b$ is 2^r.

5	**(i)(a)** $3^{20} = 3,486,784,401$ **(i)(b)** $3\times 2^{19} = 1,572,864$ **(ii)(a)** No. of ways to distribute the muffins to a student $= \binom{3}{1}+\binom{3}{2}+\binom{3}{3} = 7$ (OR $2^3 - 1$) Answer $= 7^6 = 117,649$ **(ii)(b)** No. of ways to distribute 4 identical objects (muffins) into 3 distinct boxes (flavours) $= \binom{4+2}{2} = 15$ (OR, $3+2\times\binom{3}{2}+\binom{3}{2}+3=15$) Answer $=15\times 14\times\cdots\times 10 = 3,603,600$ (OR, ${}^{15}P_6$) **(iii)** Total number of ways to throw away the muffins $= \binom{6+3}{3}\binom{7+3}{3}\binom{8+3}{3} = 1,663,200$ Let A_i be the set comprising of all possible combinations the muffins can be disposed into the bins, such that auntie did not throw any muffin into the i^{th} bin. $\lvert A_i\rvert = \binom{6+2}{2}\binom{7+2}{2}\binom{8+2}{2} = 46,110$ $\lvert A_i \cup A_j\rvert = \binom{6+1}{1}\binom{7+1}{1}\binom{8+1}{1} = 504$ (OR $7\times 8\times 9$) $\lvert A_i \cup A_j \cup A_k\rvert = \binom{6+0}{0}\binom{7+0}{0}\binom{8+0}{0} = 1$ Answer $= 1,663,200 - \binom{4}{1}\lvert A_i\rvert + \binom{4}{2}\lvert A_i\cup A_j\rvert - \binom{4}{3}\lvert A_i\cup A_j\cup A_k\rvert$ $= 1,484,780$
6	**(i)(a)** $L_{n,n} = 1$ $L_{n,1} = n!$ **(i)(b)** First, choose 2 objects from the *n* distinct objects to be “together” in a box. This gives us $\binom{n}{2}$ ways of doing do. Next, we can give ordering to the 2 chosen objects within that particular box. By multiplicative principle, we have $\binom{n}{2}\times 2! = n^2 - n$ ways

(ii)(c)

Case 1: Object A is alone in a box

We can then distribute the remaining $n-1$ objects into the remaining $k-1$ boxes. This gives us $L_{n-1,k-1}$ ways of doing so.

Case 2: Object A is not alone

First, we set aside this object A and arrange the remaining $n-1$ books into the k boxes. We have $L_{n-1,k}$ ways of doing so.

Next, we want to slot this object A amongst the $n-1$ books in the k boxes. Note that we can put this object A on top (to the left) of any of the $n-1$ objects or we can also put object A right at the bottom (or rightmost) of any of the k objects. So, total of $(n+k-1)$ ways of slotting in this object A.

By multiplicative principle, we have $(n+k-1)L_{n-1,k}$ for case 2.

Note that case 1 and case 2 are mutually exclusive. By addition principle, we conclude that

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

(ii)(a)

Where ordering within each box matters, we have $l_{n,k} = (k!)L_{n,k}$

$$\text{ie. } L_{n,k} = \frac{l_{n,k}}{k!}$$

From (iii),

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

$L_{n,k} = L_{n-1,k-1} + (n+k-1)L_{n-1,k}$

$\frac{l_{n,k}}{k!} = \frac{l_{n-1,k-1}}{(k-1)!} + (n+k-1)\frac{I_{n-1,k}}{k!}$

Therefore, $l_{n,k} = k\,l_{n-1,k-1} + (n+k-1)l_{n-1,k}$, $n \ge k \ge 2$

(ii)(b) Dividing by $n!$, we remove the permutations of distinction of objects as well as the ordering of objects within each box.

Hence $\frac{l_{n,k}}{n!}$ gives the number of ways to distribute n identical objects into k distinct boxes, such that no box is empty.

To ensure no boxes are empty, we first place one object into each of the k boxes. We then distribute the remaining $(n-k)$ objects, this is akin to arrangement $(n-k)$ zeros and $(k-1)$ ones, thus giving $\binom{(n-k)+(k-1)}{k-1}$ ways.

Therefore, $\frac{l_{n,k}}{n!} = \binom{n-1}{k-1}$.

(ii)(c) Let A be the set of all possible permutations of n distinct elements into k ordered partitions, where partitions may be empty.

For the case where $n=5$ and $k=3$, $\big((2,1),(3),(5,4)\big)$ means box 1, 2 and 3 contain objects 1 and 2, object 3, objects 4 and 5 respectively. In terms of arrangements within each boxes, in box 1, object 2 comes first before object 1; whereas in box 3, object 5 comes first before object 4.

	We can see that $\lvert A\rvert=\sum_{j=1}^{k}\binom{k}{j}l_{n,j}=n!\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1}$ which counts the number of ways to distribute n distinct objects into k distinct boxes such that the ordering of objects within each box is important and that boxes may be empty. Let B be all possible permutations of the $(n+k-1)$ numbers involving n distinct integers from $\{1,2,\cdots,n\}$, as well as $(k-1)$ zeros. We can see that there exists a bijection f between A and B. In the case where $n=5$ and $k=3$, $$\text{f}\big((2,1),(3),(5,4)\big)=2103054$$ $$\text{f}\big((1,3,5,4),(),(2)\big)=1354002$$ Now, arranging $\{1,2,\cdots,n;\underbrace{0,0,\cdots,0}_{k\text{ zeros}}\}$ gives us $\lvert B\rvert=\dfrac{(n+k-1)!}{(k-1)!}$. Since $\lvert A\rvert=\lvert B\rvert$, we have $n!\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1}=\dfrac{(n+k-1)!}{(k-1)!}$. Therefore, $\sum_{j=1}^{k}\binom{k}{j}\binom{n-1}{j-1}=\dfrac{(n+k-1)!}{n!(k-1)!}$.
7	**(i)** Let P_n be the statement $2u_n^{\ 2}-3v_n^{\ 2}+6u_nv_n=5$ for $n\in\mathbb{Z}^+$. For n = 1, $\text{LHS}=2u_1^{\ 2}-3u_1^{\ 2}+6u_1v_1=2(1)-3(1)+6=5=\text{RHS}$. So P_1 is true. Assume that P_k is true for some $k\in\mathbb{Z}^+$, i.e. $2u_k^{\ 2}-3v_k^{\ 2}+6u_kv_k=5$. LHS of $P_{k+1}=2u_{k+1}^{\ 2}-3v_{k+1}^{\ 2}+6u_{k+1}v_{k+1}$ $=2\left(u_k^{\ 2}+6u_kv_k+9v_k^{\ 2}\right)-3\left(4u_k^{\ 2}+28u_kv_k+49v_k^{\ 2}\right)+6\left(2u_k^{\ 2}+13u_kv_k+21v_k^{\ 2}\right)$ $=2u_k^{\ 2}-3v_k^{\ 2}+6u_kv_k$ $=5.$ So P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by the Principle of Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$. [**(ii)** For $2u_n^{\ 2}-3v_n^{\ 2}+6u_nv_n=5$, dividing throughout by $v_n^{\ 2}$ yields $2r_n^{\ 2}-3+6r_n=\dfrac{5}{v_n^{\ 2}}$. Since $\lim_{n\to\infty}(v_n)=\infty$ and $\lim_{n\to\infty}(r_n)=L$, $2L^2-3+6L=0 \quad\Rightarrow\quad L=\frac{1}{2}\left(-3+\sqrt{15}\right)$ or $\frac{1}{2}\left(-3-\sqrt{15}\right)$. However, as $r_n>0$, its limit L must be non-negative. Therefore, $L=\frac{1}{2}\left(-3+\sqrt{15}\right)$.

	(iii) $r_{n+1}-r_n=\dfrac{u_n+3v_n}{2u_n+7v_n}-\dfrac{u_n}{v_n}$ $=\dfrac{u_nv_n+3v_n{}^2-2u_n{}^2-7u_nv_n}{v_n(2u_n+7v_n)}$ $=-\dfrac{2u_n+6u_nv_n-3v_n{}^2}{v_n(2u_n+7v_n)}$ $=-\dfrac{5}{v_n(2u_n+7v_n)}<0$ (since $u_n,\ v_n>0$). So the sequence $\{r_n\}$ is strictly decreasing and tends to the limit $\frac{1}{2}\left(-3+\sqrt{15}\right)$. $\therefore\ r_n>\frac{1}{2}\left(-3+\sqrt{15}\right)$ $\Rightarrow 2r_n+3>\sqrt{15}$ ---- (2) $\Rightarrow \dfrac{1}{2r_n+3}<\dfrac{1}{\sqrt{15}}$ $\Rightarrow \dfrac{15}{2r_n+3}<\sqrt{15}$ ---- (3) By taking intersection of (2) and (3), we have the result. **(iv)** $r_3=\dfrac{u_3}{v_3}=\dfrac{31}{71}$. So $\dfrac{15}{2r_3+3}<\sqrt{15}<2r_3+3 \quad \Rightarrow \dfrac{213}{55}<\sqrt{15}<\dfrac{275}{71}$.
8	**(i)** $\phi(p)=p-1$ since all the numbers 1, 2, 3, …, $p-1$ are all relatively prime to the prime number p. **(ii)** The only positive integers not exceeding p^r which are not coprime to p^r are $1p,\ 2p,\ 3p,\ ...,\ p^{r-1}, p^r$. As there are p^{r-1} of such integers, $\phi\left(p^r\right)=p^r-p^{r-1}=p^{r-1}\left(p-1\right)$. **(iii)** <u>Case (1):</u> x is a power of 2 only ie $x=2^r, r\geq 2$. Then $\phi\left(x\right)=\phi\left(2^r\right)=2^{r-1}\left(2-1\right)=2^{r-1}$ which is even. <u>Case (2)</u> x is not a power of 2 only, then it has an odd prime p as a factor. Thus $x=p^rq$ where $r\geq 1$ and $\gcd\left(p^r,q\right)=1$. By the above result, $\phi\left(x\right)=\phi\left(p^rq\right)=\phi\left(p^r\right)\phi\left(q\right)=p^{r-1}\left(p-1\right)\phi\left(q\right)$ Since p is an odd prime, then $2\mid\left(p-1\right)$ ie $\phi\left(x\right)$ is even Hence $\phi\left(x\right)$ is even for all $x\geq 3$.

(iv) Consider $n = 2^t 3^s m$ where t and s are non-negative integers and m is a positive integer which does not have 2 or 3 as factors, ie $\gcd(6, m) = 1$

$$\phi(3x) = \phi(2x)$$
$$\phi\left(3.2^t 3^s m\right) = \phi\left(2.2^t 3^s m\right)$$
$$\phi\left(2^t 3^{s+1} m\right) = \phi\left(2^{t+1} 3^s m\right)$$
$$\phi\left(2^t 3^{s+1}\right)\phi(m) = \phi\left(2^{t+1} 3^s\right)\phi(m) \text{ since } m \text{ is coprime to both } 2^t 3^{s+1} \text{ and } 2^{t+1} 3^s$$
$$\phi\left(2^t 3^{s+1}\right) = \phi\left(2^{t+1} 3^s\right)$$
$$\phi\left(2^t\right)\phi\left(3^{s+1}\right) = \phi\left(2^{t+1}\right)\phi\left(3^s\right) \text{ since powers of 2's and powers of 3's are coprime}$$

<u>Case 1: For $s \ge 1, t \ge 1,$</u>

$$\phi\left(2^t\right)\phi\left(3^{s+1}\right) = \phi\left(2^{t+1}\right)\phi\left(3^s\right)$$
$$2^{t-1}(2-1)3^s(3-1) = 2^t(2-1)3^{s-1}(3-1)$$
$$2^{t-1}3^s = 2^t 3^{s-1}$$
$$3 = 2 \quad (no\ solution)$$

<u>Case 2: For $s \ge 1, t = 0,$</u>

$$\phi\left(2^t\right)\phi\left(3^{s+1}\right) = \phi\left(2^{t+1}\right)\phi\left(3^s\right)$$
$$\phi(1)\phi\left(3^{s+1}\right) = \phi(2)\phi\left(3^s\right)$$
$$3^s(3-1) = 3^{s-1}(3-1)$$
$$3 = 1 \quad (no\ solution)$$

<u>Case 3: For $s = 0, t \ge 1,$</u>

$$\phi\left(2^t\right)\phi\left(3^{s+1}\right) = \phi\left(2^{t+1}\right)\phi\left(3^s\right)$$
$$\phi\left(2^t\right)\phi(3) = \phi\left(2^{t+1}\right)\phi(1)$$
$$2^{t-1}(2-1)(2) = 2^t(2-1)(1)$$
$$2^t = 2^t$$

ie true for every $t \ge 1$ and $s = 0$ and positive integers m where $\gcd(6, m) = 1$.

<u>Case 4: For $s = 0, t = 0,$</u>

$$\phi\left(2^t\right)\phi\left(3^{s+1}\right) = \phi\left(2^{t+1}\right)\phi\left(3^s\right)$$
$$\phi(1)\phi(3) = \phi(2)\phi(1)$$
$$(1)(2) = (1)(1)$$
$$2 = 1 \quad (no\ solution)$$

Therefore the positive values of x are all even numbers which do not have 3 as a factor.