2019 Raffles Institution H2 Further Math Prelim Exam Paper 1 Solution

Qn	Suggested Solution	Comment
1(a) **[1]**	Consider $a=2$ and $n=1$, $a^{2n+2}-(a^2-1)n-a^2=2^4-3(1)-4=9$ and this is clearly not divisible by 576.	A handful of students consider the case when $a=1$, i.e. the expression will take the value of 0, however, 0 is divisible by 576.
(b) **[5]**	Let P_n be the statement $a^{2n+2}-(a^2-1)n-a^2=576m_n$, $m_n\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$. When $n=1$, $a^{2n+2}-(a^2-1)n-a^2=a^4-(a^2-1)-a^2=a^4-2a^2+1=(a^2-1)^2$ which is divisible by 576 (additional condition). Thus, P_1 is true. Assume that P_k is true for **some** $k\in\mathbb{Z}^+$. i.e. assume $a^{2k+2}-(a^2-1)k-a^2=576m_k$, $m_k\in\mathbb{Z}$ To prove that P_{k+1} is true, i.e. to prove $a^{2(k+1)+2}-(a^2-1)(k+1)-a^2=576m_{k+1}$, $m_{k+1}\in\mathbb{Z}$ Consider $a^{2(k+1)+2}-(a^2-1)(k+1)-a^2$ $=a^2(576m_k+(a^2-1)k+a^2]-(a^2-1)(k+1)-a^2$ $=576m_ka^2+(a^2-1)(a^2k-k-1)+a^2(a^2-1)$ $=576m_ka^2+(a^2-1)(a^2(k+1)-(k+1))$ $=576m_ka^2+(a^2-1)^2(k+1)$ $=576m_ka^2+576m_1(k+1)=576(m_ka^2+m_1(k+1))$ Since $m_1,m_k,a,k\in\mathbb{Z}$, $m_ka^2+m_1(k+1)\in\mathbb{Z}$. Alternatively, $\left[a^{2(k+1)+2}-(a^2-1)(k+1)-a^2\right]-\left[a^{2k+2}-(a^2-1)k-a^2\right]$ $=a^{2k+2}(a^2-1)-(a^2-1)$ $=(a^2-1)(a^{2k+2}-1)$ $=(a^2-1)(576m_k+(a^2-1)k+a^2-1)$ $=(a^2-1)(576m_k)+(a^2-1)^2(k+1)$ $=(a^2-1)(576m_k)+(576m_1)(k+1)=576((a^2-1)m_k+(k+1)m_1)$	Students are reminded to make use of the assumption made in the proof.

	Hence $\left[a^{2(k+1)+2}-(a^2-1)(k+1)-a^2\right]$ $=576((a^2-1)m_k+(k+1)m_1+m_k)=576(a^2m_k+(k+1)m_1)$ Hence P_k is true $\Rightarrow P_{k+1}$ is true and since P_1 is true, by Mathematical Induction, P_n is true for **all** $n\in\mathbb{Z}^+$. □	

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2(i) **[2]**	Using linear interpolation, an approximation to α $=\dfrac{(-1.1)\mathrm{f}(-1.3)-(-1.3)\mathrm{f}(-1.1)}{\mathrm{f}(-1.3)-\mathrm{f}(-1.1)}$ ≈ -1.138023472 $=-1.138$ (3 dp).	A few students could not recall the correct formula.
(ii) **[3]**	Using the Newton-Raphson method, $\alpha_{n+1}=\alpha_n-\dfrac{\mathrm{f}(\alpha_n)}{\mathrm{f}'(\alpha_n)}$ $=\alpha_n-\dfrac{\alpha_n-\frac{1}{2}\tan\alpha_n}{1-\frac{1}{2}\sec^2\alpha_n}$ $=\dfrac{2\alpha_n-\alpha_n\sec^2\alpha_n-2\alpha_n+\tan\alpha_n}{2-\sec^2\alpha_n}$ $=\dfrac{\tan\alpha_n-\alpha_n\sec^2\alpha_n}{2-\sec^2\alpha_n}$ $=\dfrac{\sin\alpha_n\cos\alpha_n-\alpha_n}{2\cos^2\alpha_n-1}$ (*) $=\dfrac{\frac{1}{2}\sin2\alpha_n-\alpha_n}{\cos2\alpha_n}$ $=\frac{1}{2}\tan2\alpha_n-\alpha_n\sec2\alpha_n$	Most students manage to get the proof right, though not in the most efficient way.
(iii) **[2]**	Using part **(ii)** with $\alpha_1=-1.1380235$ (7 dp), we have $\alpha_2=\frac{1}{2}\tan(2\alpha_1)-(\alpha_1)\sec(2\alpha_1)=-1.16827$ (5 dp). $\alpha_3=\frac{1}{2}\tan(2\alpha_2)-(\alpha_2)\sec(2\alpha_2)=-1.16559$ (5 dp). $\alpha_4=\frac{1}{2}\tan(2\alpha_3)-(\alpha_3)\sec(2\alpha_3)=-1.16556$ (5 dp). $\alpha_5=\frac{1}{2}\tan(2\alpha_4)-(\alpha_4)\sec(2\alpha_4)=-1.16556$ (5 dp). Hence, $\alpha=-1.16556$ (5 dp).	Some students did not give the answers to 5 d.p. For this question, students do not need to do checking for α.

Qn	Suggested Solution	Comment
3(i) **[2]**	$u_{r+1} = 2v_r - 6u_r$ --- (1) $v_{r+1} = 2v_r - 14u_r$ --- (2) (1) - (2): $u_{r+1} - v_{r+1} = 8u_r$ From (1), $u_{r+1} - \frac{1}{2}\left(u_{r+2} + 6u_{r+1}\right) = 8u_r$ $2u_{r+1} - u_{r+2} - 6u_{r+1} = 16u_r$ $u_{r+2} + 4u_{r+1} + 16u_r = 0$	
(ii) **[5]**	The characteristic equation for $u_{r+2} + 4u_{r+1} + 16u_r = 0$ is $\lambda^2 + 4\lambda + 16 = 0$ $\lambda = \frac{-4 \pm \sqrt{16 - 4(16)}}{2} = -2 \pm 2\sqrt{3}\,\mathrm{i}$ $u_r = 4^r\left(A\cos\frac{2r\pi}{3} + B\sin\frac{2r\pi}{3}\right)$ Put $r = 0$, $2 = A$ From equation (1) in part (i), $u_1 = 2(10) - 6(2) = 8$ Put $r = 1$ into u_r, $8 = 4\left(-1 + B\frac{\sqrt{3}}{2}\right) \Rightarrow B = 2\sqrt{3}$ $u_r = 4^r\left(2\cos\frac{2r\pi}{3} + 2\sqrt{3}\sin\frac{2r\pi}{3}\right)$ $= 4^r\sqrt{4+12}\cos\left(\frac{2r\pi}{3} - \frac{\pi}{3}\right)$ or $4^r\sqrt{4+12}\sin\left(\frac{2r\pi}{3} + \frac{\pi}{6}\right)$ $= 4^{r+1}\cos\frac{(2r-1)\pi}{3}$ or $4^{r+1}\sin\frac{(4r+1)\pi}{6}$	Students should note that $-2 \pm 2\sqrt{3}\,\mathrm{i} = 4\mathrm{e}^{\pm\frac{2\pi}{3}\mathrm{i}}$ A couple of students missed out the $\pm$ sign. A number of students left the answer in terms of n instead. Some of them did not use R-formula to express the answer as a single trigo function.
(iii) **[1]**	For integer r, $\cos\frac{(2r-1)\pi}{3} = \frac{1}{2}$ or -1. So $\frac{\lvert u_{r+1}\rvert}{\lvert u_r\rvert} = 4\left\lvert\frac{\cos\frac{(2r+1)\pi}{3}}{\cos\frac{(2r-1)\pi}{3}}\right\rvert \geq 4\left(\frac{1}{2}\right) = 2 \Rightarrow \lvert u_{r+1}\rvert \geq 2\lvert u_r\rvert$ Thus, the sequence is strictly increasing.	The possible values of $\cos\frac{(2r-1)\pi}{3}$ could be observed from GC.

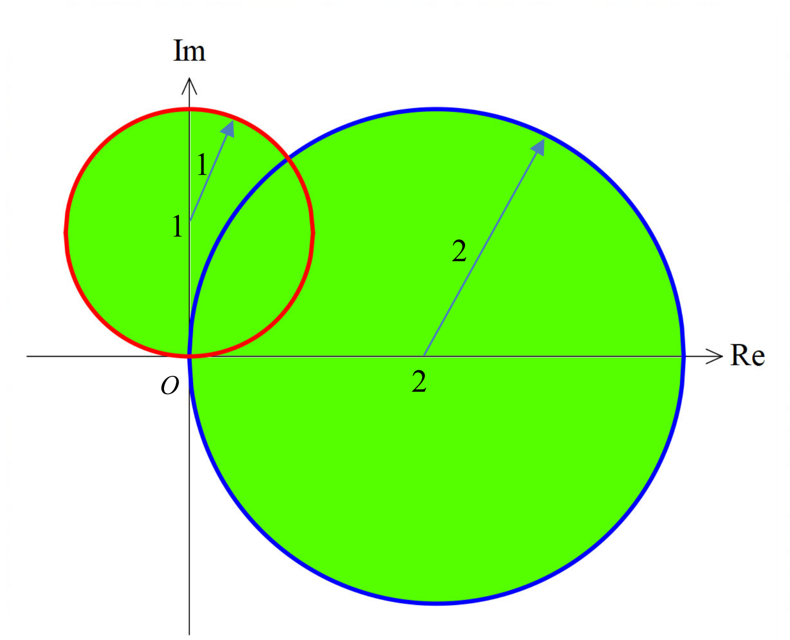
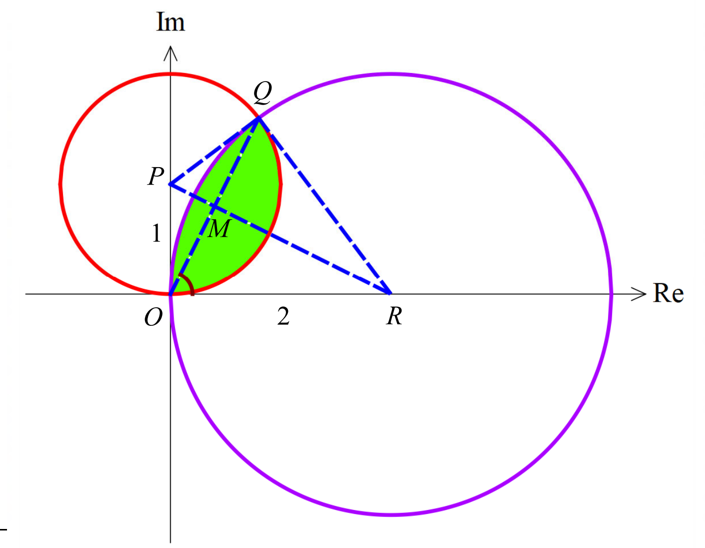
Qn	Suggested Solution	Comment
4(i) **[1]**	For $a \in \mathbb{R}$, $$\begin{aligned}\mathbf{A}\left(\mathbf{x}_1 + a\mathbf{x}_2\right) &= \mathbf{A}\mathbf{x}_1 + \mathbf{A}\left(a\mathbf{x}_2\right) \\ &= \mathbf{A}\mathbf{x}_1 + a\mathbf{A}\mathbf{x}_2 \\ &= \mathbf{b}_1 + a\mathbf{b}_2 \quad \text{(Shown).}\end{aligned}$$	This question is well done.
(ii) **[3]**	Using GC, $$\mathbf{A}\mathbf{x}_1 = \mathbf{b}_1 \Rightarrow \mathbf{x}_1 = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}, \quad \mathbf{A}\mathbf{x}_2 = \mathbf{b}_2 \Rightarrow \mathbf{x}_2 = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix} \text{ and}$$ $$\mathbf{A}\mathbf{x}_3 = \mathbf{b}_1 \Rightarrow \mathbf{x}_3 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix}.$$	
(iii) **[3]**	Using part **(i)**, we have $$\mathbf{A}\left(\mathbf{x}_2 - 2\mathbf{x}_1\right) = \mathbf{b}_2 - 2\mathbf{b}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$ so the solution is $\mathbf{x}_2 - 2\mathbf{x}_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{3}{5} \\ -\frac{2}{5} \end{pmatrix} - 2\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \end{pmatrix}$. Similarly, we have $$\mathbf{A}\left[\mathbf{x}_3 + 3\left(\mathbf{x}_2 - 2\mathbf{x}_1\right)\right] = \mathbf{b}_3 + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$ so the solution is $\mathbf{x}_3 + 3\left(\mathbf{x}_2 - 2\mathbf{x}_1\right) = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ \frac{5}{2} \end{pmatrix} + 3\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ \frac{1}{10} \\ \frac{1}{10} \end{pmatrix}$.	
(iv) **[2]**	Since $\mathbf{A}\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}$ is the first column of $\mathbf{A}^{-1}$. Similarly, $\mathbf{A}\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{4}{5} \end{pmatrix}$ is the second column of $\mathbf{A}^{-1}$ and $\mathbf{A}\begin{pmatrix} -\frac{2}{5} \\ \frac{1}{10} \\ \frac{1}{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} -\frac{2}{5} \\ \frac{1}{10} \\ \frac{1}{10} \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -\frac{2}{5} \\ \frac{1}{10} \\ \frac{1}{10} \end{pmatrix}$ is the third column of $\mathbf{A}^{-1}$. Hence, $\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 2 & 2 & -4 \\ 2 & 2 & 1 \\ 2 & -8 & 1 \end{pmatrix}$, where $k = 10$.	

Qn	Suggested Solution	Comment
5(i) **[4]**	Consider the intersection of the line $y = mx + d$ and the ellipse, the x-coordinates of the two points on the end of the chord satisfy the equation $\frac{x^2}{a^2} + \frac{(mx+d)^2}{b^2} = 1$ Simplifying, we get $\left(a^2m^2 + b^2\right)x^2 + 2a^2mdx + a^2\left(d^2 - b^2\right) = 0$ Let the midpoint be (x_M, y_M), we have $x_M = \frac{-\frac{2a^2md}{a^2m^2+b^2}}{2} = -\frac{a^2md}{a^2m^2+b^2}$ $y_M = m\left(-\frac{a^2md}{a^2m^2+b^2}\right) + d = \frac{-a^2m^2d + a^2m^2d + b^2d}{a^2m^2+b^2} = \frac{b^2d}{a^2m^2+b^2}$ So $(x_M, y_M) = \left(-\frac{a^2md}{a^2m^2+b^2}, \frac{b^2d}{a^2m^2+b^2}\right)$. (shown).	Most students applied the Vieta's formula to find x_M, while some solved for the two roots before finding the midpoint. Some solved for y_M by equating the equations of line and ellipse again, instead of using $y_M = mx_M + d$.
(ii) **[2]**	By the definition and part **(i)**, the midpoints have coordinates $\left(-\frac{a^2md}{a^2m^2+b^2}, \frac{b^2d}{a^2m^2+b^2}\right)$. $x = -\frac{a^2md}{a^2m^2+b^2}, y = \frac{b^2d}{a^2m^2+b^2}$, so by eliminating d, we have $\frac{y}{x} = \frac{b^2d}{-a^2md} = \frac{b^2}{-a^2m}$, hence $y = -\frac{b^2}{a^2m}x$. (shown).	This part was not well done. Many students assumed that the locus is a straight line in their proofs.
(iii) **[3]**	Given $c = 2\sqrt{3}$ and $e = \frac{\sqrt{3}}{2}$, we have $b = \frac{c}{e} = \frac{2\sqrt{3}}{\frac{\sqrt{3}}{2}} = 4.$ Using $c^2 = b^2 - a^2$, we have $a = \sqrt{4^2 - \left(2\sqrt{3}\right)^2} = 2$ Hence the equation is $y = -\frac{4^2}{2^2(5)}x = -\frac{4}{5}x.$	Majority of students failed to check the orientation of the ellipse and used $c = ea$ instead of $c = eb$, leading to wrong answer.

Qn	Suggested Solution	Comment
6(i) **[2]**	$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ $\mathbf{A}^n\mathbf{x} = \mathbf{A}^{n-1}(\mathbf{A}\mathbf{x})$ $= \mathrm{A}^{n-1}(\lambda \mathrm{x})$ $= \lambda \mathrm{A}^{n-1}\mathrm{x}$ $= \ldots$ $= \lambda^n \mathrm{x}$	This question is well done.

	$\left(\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\mathbf{A}^4\right)\mathbf{x} = \mathrm{A}\mathbf{x}+\mathrm{A}^2\mathbf{x}+\mathrm{A}^3\mathbf{x}+\mathrm{A}^4\mathbf{x}$ $= \lambda\mathbf{x}+\lambda^2\mathbf{x}+\lambda^3\mathbf{x}+\lambda^4\mathbf{x}$ $= \left(\lambda+\lambda^2+\lambda^3+\lambda^4\right)\mathbf{x}$	
(ii) **[6]**	$\mathbf{A}-\lambda\mathbf{I} = \begin{pmatrix}1&3&4\\0&2&8\\0&0&-3\end{pmatrix} - \lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1-\lambda&3&4\\0&2-\lambda&8\\0&0&-3-\lambda\end{pmatrix}$ The characteristic equation of **A** is det(**A**–λ**I**)=0 $\begin{vmatrix}1-\lambda&3&4\\0&2-\lambda&8\\0&0&-3-\lambda\end{vmatrix} = 0$ $\Rightarrow (1-\lambda)(2-\lambda)(-3-\lambda)=0$ $\Rightarrow \lambda = 1,2,-3$ The eigenvalues of **A** are 1, 2, and –3. $\left(\mathbf{A}-\lambda\mathbf{I}\right)\mathbf{x} = \begin{pmatrix}1-\lambda&3&4\\0&2-\lambda&8\\0&0&-3-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ $\begin{pmatrix}\left(1-\lambda\right)x_1+3x_2+4x_3\\\left(2-\lambda\right)x_2+8x_3\\\left(-3-\lambda\right)x_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ Thus, $\left(1-\lambda\right)x_1+3x_2+4x_3=0$ $\left(2-\lambda\right)x_2+8x_3=0$ $\left(-3-\lambda\right)x_3=0$ When λ=1, $3x_2+4x_3=0$ $2x_2+8x_3=0$ $-4x_3=0$ An eigenvector corresponding to the eigenvalue λ=1 is $\begin{pmatrix}1\\0\\0\end{pmatrix}$. When λ = 2, $-x_1+3x_2+4x_3=0$ $8x_3=0$ $-5x_3=0$ An eigenvector corresponding to the eigenvalue λ=2 is $\begin{pmatrix}3\\1\\0\end{pmatrix}$. When λ = –3, $4x_1+3x_2+4x_3=0$ $5x_2+8x_3=0$ An eigenvector corresponding to the eigenvalue λ=–3 is $\begin{pmatrix}1\\-8\\5\end{pmatrix}$.	

	By result in part **(i)**, the eigenvalues are 4, 30, and 60 with corresponding eigenvectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}3\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-8\\5\end{pmatrix}$ respectively.	
(iii) **[2]**	$\mathbf{D}=\begin{pmatrix}4&0&0\\0&30&0\\0&0&60\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}1&3&1\\0&1&-8\\0&0&5\end{pmatrix}$	The question explicitly asks for the matrix **Q** and matrix **D**. So, one should not leave your answer as $\mathbf{QDQ}^{-1}$

Qn	**Suggested Solution**	**Comment**
7(i) **[2]**	The set B contains all points in the interior (and the boundary) of the circle centered at $(b, 1)$ with radius 1. $b = 2$.	
(ii) **[2]**		The loci involve inequalities, and so, you will need to shade the regions. For locus that is a circle, do use a pair of compasses to draw and it is important to state the coordinates of the centre and radius of the circle.
(iii) **[5]**	<u>**Geometrical Method**</u> 	Most did this part using the algebraic method.

Quadrilateral $OPQR$ is a kite such that $OP = PQ = 1$, $OR = QR = 2$, and $\angle POR = \angle OMR = \angle PQR = 90^\circ$, where M is the point where the diagonals PR and OQ intersect.

$\angle QOR = 90^\circ - \angle POQ = \angle OPM$.

Thus, the value of $\arg z = \angle QOR = \angle OPR = \tan^{-1}\dfrac{OR}{OP} = \tan^{-1} 2$.

$$OM = OR\cos\angle MOR = 2\left(\frac{1}{\sqrt{5}}\right) = \frac{2}{\sqrt{5}}$$

Then, $z = OQ\left(\cos\angle QOR + \mathrm{i}\sin\angle QOR\right)$

$$= \frac{4}{\sqrt{5}}\left(\frac{1}{\sqrt{5}} + \mathrm{i}\frac{2}{\sqrt{5}}\right) = \frac{4}{5} + \frac{8}{5}\mathrm{i}$$

<u>Algebraic Method</u>

Solving for the two points of intersection, from the 2 circles we have

$(x-2)^2 + y^2 = 4,\ x^2 + (y-1)^2 = 1.$

Thus we have

$(x-2)^2 + y^2 - 4 = x^2 + (y-1)^2 - 1$

$\Rightarrow x^2 - 4x + y^2 = x^2 + y^2 - 2y$

$\Rightarrow y = 2x$

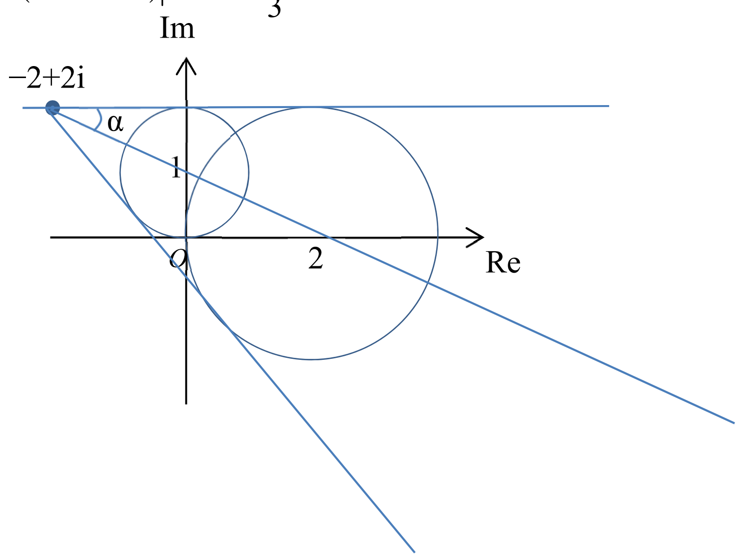
Hence maximum $\arg z = \tan^{-1} 2$.

Substituting this into the original equation, we have $5x^2 - 4x = 0 \Rightarrow x = \dfrac{4}{5}$. Hence $z = \dfrac{4}{5} + \dfrac{8}{5}\mathrm{i}$.

(iv)
[3]

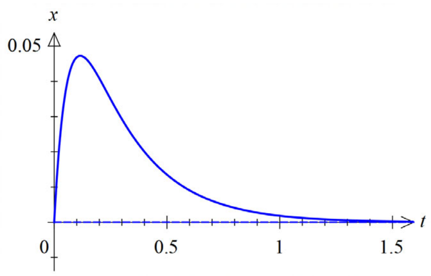
We note that the points representing the complex numbers $-2 + 2\mathrm{i}$, i and 2 are collinear. In addition, the point representing the complex number $-2 + 2\mathrm{i}$ lies on the common tangent of the 2 circles. Thus the maximum $\left|\arg\left(z + 2 - 2\mathrm{i}\right)\right|$ occurs on the other common tangent to both circles. We have $\tan\alpha = \dfrac{1}{2} \Rightarrow d = \tan 2\alpha = \dfrac{1}{1-\frac{1}{4}} = \dfrac{4}{3}$. Therefore,

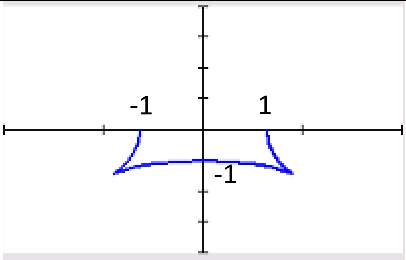
$$\max\left|\arg\left(z + 2 - 2\mathrm{i}\right)\right| = \tan^{-1}\frac{4}{3}.$$



Many thought the angle is found using the line that passes through the pole.

Qn	Suggested Solution	Comment
8(i) **[3]**	$\sqrt{(x-c)^2+y^2}+vt=\sqrt{(x+c)^2+y^2}$ $(x-c)^2+y^2+2vt\sqrt{(x-c)^2+y^2}+v^2t^2=(x+c)^2+y^2$ $4v^2t^2\left(x^2-2cx+c^2+y^2\right)=v^4t^4-8cv^2t^2x+16c^2x^2$ $4x^2\left(v^2t^2-4c^2\right)+4y^2\left(v^2t^2\right)=v^2t^2\left(v^2t^2-4c^2\right)$ $\dfrac{4x^2}{v^2t^2}-\dfrac{4y^2}{4c^2-v^2t^2}=1$ (shown)	Students to note that they are required to show that (x,y) lies on a hyperbola. Those who used properties of hyperbola in their proofs were not awarded.
(ii) **[3]**	$c=75,\ e=\dfrac{5}{3}\ \Rightarrow a=\dfrac{c}{e}=45$ Since $a^2=\dfrac{v^2t^2}{4}\quad\Rightarrow\dfrac{(300{,}000)^2t^2}{4}=45^2$ So $t=0.0003$.	This part was well done.
(iii) **[3]**	$p=ea-\dfrac{a}{e}=\dfrac{5}{3}(45)-\dfrac{45}{\left(\frac{5}{3}\right)}=48$, so $ep=\dfrac{5}{3}(48)=80$ $H_1: r=\dfrac{80}{1-\frac{5}{3}\cos\theta}\qquad H_2: r=\dfrac{80}{1+\frac{5}{3}\cos\left(\theta+\frac{\pi}{4}\right)}$ (or $r=\dfrac{240}{3-5\cos\left(\theta-\frac{3\pi}{4}\right)}$)	Some common mistakes seen in equation of H_2 were $r=\dfrac{80}{1+\frac{5}{3}\cos\left(\theta-\frac{\pi}{4}\right)}$ and $r=\dfrac{80}{1-\frac{5}{3}\cos\left(\theta+\frac{\pi}{4}\right)}$. Students should use GC to double check their graphs.
(iv) **[3]**	To find the intersection between H_1 and H_2, $\dfrac{80}{1-\frac{5}{3}\cos\theta}=\dfrac{80}{1+\frac{5}{3}\cos\left(\theta+\frac{\pi}{4}\right)}$ $\dfrac{5}{3}\left[\cos\left(\theta+\dfrac{\pi}{4}\right)+\cos\theta\right]=0$ $\dfrac{5}{3}\left[2\cos\left(\theta+\dfrac{\pi}{8}\right)\cos\dfrac{\pi}{8}\right]=0$ by factor formula $\cos\left(\theta+\dfrac{\pi}{8}\right)=0$ $\theta+\dfrac{\pi}{8}=\dfrac{\pi}{2}$ or $\theta+\dfrac{\pi}{8}=\dfrac{3\pi}{2}$ $\theta=\dfrac{3\pi}{8}$ or $\theta=\dfrac{11\pi}{8}$ (rej. since this lies in $[\pi,2\pi]$) When $\theta=\dfrac{3\pi}{8}$, $r=220.876$ (to 6s.f.) Hence polar coordinates of the location of the ship is $\left(220.876,\dfrac{3\pi}{8}\right)$.	Most who arrived here were able to simplify the equation using either factor formula or addition formula, although some did not account for $\theta\neq\dfrac{11\pi}{8}$ before concluding the final value of θ.

Qn	Suggested Solution	Comment
9 [7]	Differential equation: $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+20\frac{\mathrm{d}x}{\mathrm{d}t}+64x=0$ Auxiliary equation: $\lambda^2+20\lambda+64=0$ $\lambda=-4$ or $\lambda=-16$ General solution: $x=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{-16t}$ When $t=0$, $x=0$, then $c_1+c_2=0$ $c_1=-c_2$ $x=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{-16t} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=-4c_1\mathrm{e}^{-4t}-16c_2\mathrm{e}^{-16t}$ When $t=0$, $\frac{\mathrm{d}x}{\mathrm{d}t}=1.2$, then $1.2=-4c_1-16c_2$ $1.2=-4(-c_2)-16c_2$ $-12c_2=1.2$ $c_2=-0.1$ $c_1=0.1$ So, $x=0.1\mathrm{e}^{-4t}-0.1\mathrm{e}^{-16t}=0.1\left(\mathrm{e}^{-4t}-\mathrm{e}^{-16t}\right)$	Most students were able to tackle this question.
[2]	$x=0.1\left(\mathrm{e}^{-4t}-\mathrm{e}^{-16t}\right)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=0.1\left(-4\mathrm{e}^{-4t}+16\mathrm{e}^{-16t}\right)$ The particle changes direction when $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ $-4\mathrm{e}^{-4t}+16\mathrm{e}^{-16t}=0$ $\mathrm{e}^{12t}=4$ $12t=\ln 4$ $t=\frac{1}{12}\ln 4=\frac{1}{6}\ln 2$ The particle changes direction at $t=\frac{1}{6}\ln 2$ s	Students are reminded to give the answer in exact form.
[3]	 P moves away from its equilibrium position quickly and at $t=\frac{1}{6}\ln 2$ s, P changes direction and moves back towards the equilibrium position.	A handful of students lack detail in the description of the motion of P.

Qn	Suggested Solution	Comment
10(i) **[1]**	 	Symmetry and/or labelling is not observed in a number of scripts.
(ii) **[2]**	$x=(1+2\sin^2\theta)\cos\theta$ $\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=-\sin\theta(1+2\sin^2\theta)+\cos\theta(4\sin\theta\cos\theta)$ $=-\sin\theta-2\sin^3\theta+4\sin\theta(1-\sin^2\theta)$ $=3\sin\theta-6\sin^3\theta$ $=3\sin\theta(1-2\sin^2\theta)=3\sin\theta\cos 2\theta$	
(iii) **[5]**	$y=(-3+2\sin^2\theta)\sin\theta=-3\sin\theta+2\sin^3\theta$ $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=-3\cos\theta+6\sin^2\theta\cos\theta$ $=3\cos\theta(2\sin^2\theta-1)=-3\cos\theta\cos 2\theta$ $\left(\dfrac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2$ $=9\sin^2\theta(\cos 2\theta)^2+9\cos^2\theta(\cos 2\theta)^2$ $=9(\cos 2\theta)^2$ Distance travelled $=\displaystyle\int_0^{\frac{\pi}{6}}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $=\displaystyle\int_0^{\frac{\pi}{6}} 3\lvert\cos 2\theta\rvert\ \mathrm{d}\theta$ $=3\displaystyle\int_0^{\frac{\pi}{6}} \cos 2\theta\ \mathrm{d}\theta$ $=3\left[\dfrac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$ $=\dfrac{3\sqrt{3}}{4}$	Important to note: $\displaystyle\int_0^{\frac{\pi}{6}}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $=\displaystyle\int_0^{\frac{\pi}{6}} 3\lvert\cos 2\theta\rvert\ \mathrm{d}\theta$ In this case, for $0\le\theta\le\dfrac{\pi}{6}$, $\cos 2\theta>0$, thus $\displaystyle\int_0^{\frac{\pi}{6}} 3\lvert\cos 2\theta\rvert\ \mathrm{d}\theta$ $=3\displaystyle\int_0^{\frac{\pi}{6}} \cos 2\theta\ \mathrm{d}\theta$
(iv) **[3]**	Surface area $=\displaystyle\int_0^{\frac{\pi}{2}} 2\pi x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\ \mathrm{d}\theta$ $=\displaystyle\int_0^{\frac{\pi}{2}} 2\pi\cos\theta(1+2\sin^2\theta)\lvert 3\cos 2\theta\rvert\ \mathrm{d}\theta$ $=17.6\,\text{unit}^2$ (3s.f.)	Some students could not recall the correct formula. In this question, the modulus is crucial.

(v) **[3]**	Volume of the solid $= 2\left(\pi\int_0^{\sqrt{2}} y^2 \ \mathrm{d}x - \pi\int_1^{\sqrt{2}} y^2 \ \mathrm{d}x\right)$ $= 2\left(\pi\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} y^2 \frac{\mathrm{d}x}{\mathrm{d}\theta} \ \mathrm{d}\theta - \pi\int_0^{\frac{\pi}{4}} y^2 \frac{\mathrm{d}x}{\mathrm{d}\theta} \ \mathrm{d}\theta\right)$ $= 2\left[\pi\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \left((-3+2\sin^2\theta)\sin\theta\right)^2 3\sin\theta\cos 2\theta \ \mathrm{d}\theta - \pi\int_0^{\frac{\pi}{4}} \left((-3+2\sin^2\theta)\sin\theta\right)^2 3\sin\theta\cos 2\theta \ \mathrm{d}\theta\right]$ $\approx 2\pi\left(1.80481 - 0.45243\right)$ ≈ 8.4973 $= 8.50 \, \text{units}^3 \ (3\text{s.f.})$	Students should note that the interval has to be split to be able to find the required volume.

2019 Raffles Institution H2 Further Math Prelim Exam Paper 2 Solution

Qn	Suggested Solution	Comment
1 **[6]**	$C=\sum_{k=1}^{n}\cos^2(2k-1)\theta$ $=\frac{1}{2}\sum_{k=1}^{n}\left(1+\cos(4k-2)\theta\right)$ $=\frac{n}{2}+\frac{1}{2}\mathrm{Re}\sum_{k=1}^{n}\mathrm{e}^{(4k-2)\mathrm{i}\theta}$ $=\frac{n}{2}+\frac{1}{2}\mathrm{Re}\frac{\mathrm{e}^{2\mathrm{i}\theta}(\mathrm{e}^{4n\mathrm{i}\theta}-1)}{\mathrm{e}^{4\mathrm{i}\theta}-1}$ $=\frac{n}{2}+\frac{1}{2}\mathrm{Re}\frac{\mathrm{e}^{4n\mathrm{i}\theta}-1}{\mathrm{e}^{2\mathrm{i}\theta}-\mathrm{e}^{-2\mathrm{i}\theta}}$ $=\frac{n}{2}+\frac{1}{2}\mathrm{Re}\frac{\cos 4n\theta+\mathrm{i}\sin 4n\theta-1}{2\mathrm{i}\sin 2\theta}$ $=\frac{n}{2}+\frac{\sin 4n\theta}{4\sin 2\theta}$ (shown) Note that $\cos^2 A+\sin^2 A=1$, hence $C+S=n$ $\Rightarrow S=\frac{n}{2}-\frac{\sin 4n\theta}{4\sin 2\theta}$	There are a few who not able to start this question. Note that this is a "show" question. So, do not skip step in your proof.
	<u>Alternative solution:</u> Let P_n be the statement: $\sum_{r=1}^{n}\cos^2(2r-1)\theta=\frac{n}{2}+\frac{\sin 4n\theta}{4\sin 2\theta}$ where n is a positive integer. When $n=1$: $\frac{n}{2}+\frac{\sin 4n\theta}{4\sin 2\theta}=\frac{1}{2}+\frac{\sin 4\theta}{4\sin 2\theta}=\frac{1}{2}+\frac{\cos 2\theta}{2}=\cos^2\theta$ $\therefore P_1$ is true Assume that P_k is true for some $k\in\mathbb{Z}^+$, When $n=k+1$: $\sum_{r=1}^{k+1}\cos^2(2r-1)\theta$ $=\frac{k}{2}+\frac{\sin 4k\theta}{4\sin 2\theta}+\cos^2(2k+1)\theta$ $=\frac{k}{2}+\frac{\sin 4k\theta}{4\sin 2\theta}+\frac{1}{2}+\frac{\cos 2(2k+1)\theta}{2}$ $=\frac{k+1}{2}+\frac{\sin 4k\theta}{4\sin 2\theta}+\frac{2\cos(4k+2)\theta\sin 2\theta}{4\sin 2\theta}$	

	$=\dfrac{k+1}{2}+\dfrac{\sin 4k\theta}{4\sin 2\theta}+\dfrac{\sin(4k+4)\theta-\sin 4k\theta}{4\sin 2\theta}$ $=\dfrac{k+1}{2}+\dfrac{\sin 4(k+1)\theta}{4\sin 2\theta}$ $\therefore \mathrm{P}_{k+1}$ is true if P_k is true Hence P_n is true for all positive integer n, i.e. $C=\dfrac{n}{2}+\dfrac{\sin 4n\theta}{4\sin 2\theta}$ Note that $\cos^2 A+\sin^2 A=1$, hence $C+S=n$ $\Rightarrow S=\dfrac{n}{2}-\dfrac{\sin 4n\theta}{4\sin 2\theta}$	

Qn	Suggested Solution	Comment
2(i) **[2]**	Using GC, $\begin{pmatrix} 1 & 2 & 2 & 0 \\ -2 & 1 & 0 & -4 \\ 1 & -2 & 2 & 5 \\ 0 & 1 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{11}{8} \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{9}{16} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\therefore$ rank of $\mathbf{A}=3$.	This question is well done.
(ii) **[1]**	Basis of column space of $\mathbf{A}=\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 4 \end{pmatrix}\right\}$	
(iii) **[2]**	Consider $\begin{pmatrix} 1 & 2 & 2 & 0 \\ -2 & 1 & 0 & -4 \\ 1 & -2 & 2 & 5 \\ 0 & 1 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 & \frac{11}{8} \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & \frac{9}{16} \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	

	$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} -\frac{11}{8}t \\ \frac{5}{4}t \\ -\frac{9}{16}t \\ t \end{pmatrix} = t\begin{pmatrix} -\frac{11}{8} \\ \frac{5}{4} \\ -\frac{9}{16} \\ 1 \end{pmatrix}$ $\therefore$ Basis of null space $= \left\{ \begin{pmatrix} -\frac{11}{8} \\ \frac{5}{4} \\ -\frac{9}{16} \\ 1 \end{pmatrix} \right\}$	
(iv) **[3]**	$\begin{pmatrix} 1 & 2 & 2 & 0 \\ -2 & 1 & 0 & -4 \\ 1 & -2 & 2 & 5 \\ 0 & 1 & 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \\ 8 \end{pmatrix}$ Using GC, $\mathbf{x} = \begin{pmatrix} -\frac{21}{8} \\ -\frac{1}{4} \\ \frac{33}{16} \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{11}{8} \\ \frac{5}{4} \\ -\frac{9}{16} \\ 1 \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \\ -1 \end{pmatrix}$ So $t=-1 \Rightarrow p=-\frac{5}{4}, \quad q=-\frac{3}{2}, \quad r=\frac{21}{8}$.	

Qn	**Suggested Solution**	**Comment**
3(i) **[2]**	$y_{n+1} = \frac{1}{3}\left(y_n + \frac{2}{y_n}\right)$ As $n \to \infty$, $x_n \to L$. $L = \frac{1}{3}\left(L + \frac{2}{L}\right) \Rightarrow \quad 2L^2 = 2$ $L=1$ since $y_n > 0$ for all n.	This part is well done.
(ii) **[6]**	$y_{n+1} - L = \frac{1}{3}\left(y_n + \frac{2}{y_n}\right) - L = \frac{1}{3y_n}\left(y_n^2 - 3Ly_n + 2\right)$ $= \frac{1}{3y_n}\left(y_n^2 - 3y_n + 2\right)$, since $L=1$ $= \frac{1}{3y_n}\left(y_n - 2\right)\left(y_n - 1\right) < 0$ since $1 = L < y_n < 2$ Therefore $y_{n+1} < L$ for $y_n > L$. $y_{n+2} - L = \frac{1}{3y_{n+1}}\left(y_{n+1} - 2\right)\left(y_{n+1} - 1\right) > 0$ since $0 < y_{n+1} < L = 1$ Therefore $y_{n+2} > L$.	

To prove that $y_{n+2} < y_n$.

Method 1

$$y_{n+2} - y_n$$
$$= \frac{1}{3}\left(y_{n+1} + \frac{2}{y_{n+1}}\right) - y_n = \frac{1}{3}\left[\frac{1}{3}\left(y_n + \frac{2}{y_n}\right) + \frac{2}{\frac{1}{3}\left(y_n + \frac{2}{y_n}\right)}\right] - y_n$$
$$= \frac{1}{9}\left(\frac{y_n^2 + 2}{y_n}\right) + \frac{2y_n}{y_n^2 + 2} - y_n$$
$$= \frac{\left(y_n^2 + 2\right)^2 + 18y_n^2 - 9y_n^2\left(y_n^2 + 2\right)}{9y_n\left(y_n^2 + 2\right)}$$
$$= \frac{-8y_n^4 + 4y_n^2 + 4}{9y_n\left(y_n^2 + 2\right)} = -4\frac{\left(2y_n^2 + 1\right)\left(y_n^2 - 1\right)}{9y_n\left(y_n^2 + 2\right)} < 0, \text{ since } y_n > 1$$

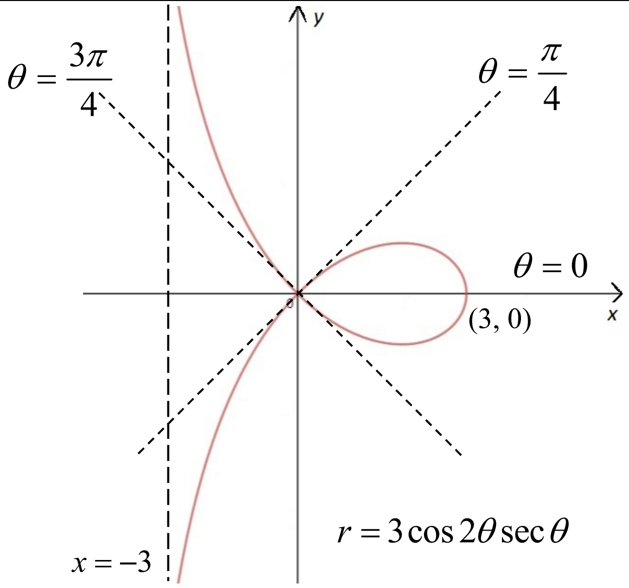
Therefore $y_{n+2} < y_n$.

Method 2

$$y_{n+2} - y_n$$
$$= y_{n+2} - y_{n+1} + y_{n+1} - y_n$$
$$= \frac{1}{3}\left(y_{n+1} + \frac{2}{y_{n+1}}\right) - \frac{1}{3}\left(y_n + \frac{2}{y_n}\right) + y_{n+1} - y_n$$
$$= \frac{4}{3}\left(y_{n+1} - y_n\right) + \frac{2}{3}\left(\frac{1}{y_{n+1}} - \frac{1}{y_n}\right) = \frac{4}{3}\left(y_{n+1} - y_n\right) + \frac{2}{3}\left(\frac{y_n - y_{n+1}}{y_n y_{n+1}}\right)$$
$$= \frac{2}{3}\left(y_{n+1} - y_n\right)\left(2 - \frac{1}{y_n y_{n+1}}\right) = \frac{2}{3}\left(y_{n+1} - y_n\right)\left(2 - \frac{1}{\frac{y_n}{3}\left(y_n + \frac{2}{y_n}\right)}\right)$$
$$= \frac{2}{3}\left(y_{n+1} - y_n\right)\left(2 - \frac{3}{y_n^2 + 2}\right)$$
$$= \frac{2}{3}\left(y_{n+1} - y_n\right)\left(\frac{2y_n^2 + 1}{y_n^2 + 2}\right) < 0, \text{since } y_{n+1} < L < y_n$$

Therefore $y_{n+2} < y_n$.

So, $y_{n+1} < L < y_{n+2} < y_n$

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th>Comment</th></tr>
<tr><td>4(i)
[3]</td><td>
</td><td>A handful of students were penalized for not including the two tangent lines. Note that it can be seen from the GC that there is a vertical asymptote located at $x = -3$.</td></tr>
<tr><td>(ii)
[4]</td><td>At the pole, $\cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$. The curve is also symmetrical about the initial line.
$\text{Area} = 2\int_0^{\frac{\pi}{4}} \frac{1}{2}(3\cos 2\theta \sec\theta)^2 \, d\theta = 9\int_0^{\frac{\pi}{4}} \cos^2 2\theta \sec^2\theta d\theta$
$= 9\int_0^{\frac{\pi}{4}} (2\cos\theta - \sec\theta)^2 d\theta = 9\int_0^{\frac{\pi}{4}} 4\cos^2\theta - 4 + \sec^2\theta d\theta$
$= 9\int_0^{\frac{\pi}{4}} 2\cos 2\theta - 2 + \sec^2\theta d\theta = 9[\sin 2\theta - 2\theta + \tan\theta]_0^{\frac{\pi}{4}}$
$= \frac{9}{2}(4 - \pi).$</td><td>This part was well done.</td></tr>
<tr><td>(iii)
[3]</td><td>$r = 3\cos 2\theta \sec\theta$
$\frac{dr}{d\theta} = 3(-2\sin 2\theta)\sec\theta + 3\cos 2\theta \sec\theta \tan\theta$
$= 3\sec\theta(\cos 2\theta \tan\theta - 2\sin 2\theta)$
$= 3\sec\theta(2\sin\theta\cos\theta - \tan\theta - 2\sin 2\theta)$
$= -3\sec\theta(\sin 2\theta + \tan\theta)$
Arc length $= 2\int_0^{\frac{\pi}{4}} \sqrt{(3\cos 2\theta \sec\theta)^2 + [-3\sec\theta(\sin 2\theta + \tan\theta)]^2} \, d\theta$
$= 6\int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta\left(\cos^2 2\theta + (\sin 2\theta + \tan\theta)^2\right)} d\theta$
$= 6\int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta\left(1 + 2\sin 2\theta \tan\theta + \tan^2\theta\right)} d\theta$
$= 6\int_0^{\frac{\pi}{4}} \sqrt{\sec^2\theta\left(\sec^2\theta + 4\sin^2\theta\right)} d\theta = 7.47$ (3.s.f)</td><td>This part was well done.</td></tr>
</table>

(iv) **[4]**	Cartesian equation of $r = 3\cos 2\theta \sec\theta$ $r\cos\theta = 3\cos 2\theta$ $x = 3\left(2\left(\frac{x}{r}\right)^2 - 1\right) = \frac{6x^2}{r^2} - 3$ $xr^2 = 6x^2 - 3r^2$ $r^2(x+3) = 6x^2$ $x^2 + y^2 = \frac{6x^2}{x+3}$ $y^2 = \frac{6x^2}{x+3} - x^2$ $\overline{x} = \frac{\int_a^b xy^2 \text{ d}x}{\int_a^b y^2 \text{ d}x}$ $= \frac{\int_0^3 \frac{6x^3}{x+3} - x^3 \text{ d}x}{\int_0^3 \frac{6x^2}{x+3} - x^2 \text{ d}x} = \frac{2.46016}{1.42995} = 1.72 \text{ (3.s.f)}$	This part was well done.

Qn	**Suggested Solution**	**Comment**
5(i) **[4]**	$\frac{\text{d}y}{\text{d}t} = t - ty \Rightarrow \int \frac{1}{1-y} \text{ d}y = \int t \text{ d}t$ $\Rightarrow -\ln\lvert 1-y\rvert = \frac{t^2}{2} + C$, where C is an arbitratry constant $\Rightarrow \ln\lvert 1-y\rvert = -\frac{t^2}{2} - C$ $\Rightarrow 1 - y = A\text{e}^{-\frac{t^2}{2}}$, where $A = \pm\text{e}^{-C}$ When $t = 0$, $y = 0$, we have $A = 1$. Hence, $y = 1 - \text{e}^{-\frac{t^2}{2}}$. When $t = 1$, $y = 0.39347$ (5 sf). <u>Alternatively</u>, $\frac{\text{d}y}{\text{d}t} = t - ty \Rightarrow \frac{\text{d}y}{\text{d}t} + ty = t$. Integrating factor, $\text{e}^{\int t \text{ d}t} = \text{e}^{\frac{t^2}{2}}$. So $y\text{e}^{\frac{t^2}{2}} = \int t\text{e}^{\frac{t^2}{2}} \text{ d}t = \text{e}^{\frac{t^2}{2}} + C$, where C is an arbitratry constant. When $t = 0$, $y = 0$, we have $C = -1$. Hence, $y\text{e}^{\frac{t^2}{2}} = \text{e}^{\frac{t^2}{2}} - 1 \Rightarrow y = 1 - \text{e}^{-\frac{t^2}{2}}$.	There were still minor mistakes seen, especially in not accounting for the modulus in $\ln\lvert 1-y\rvert$.

(ii) **[4]**	$\frac{\mathrm{d}y}{\mathrm{d}t} = t - ty$. Let $\mathrm{f}(t, y) = t - ty$. Improved Euler method with step size h: $u_{n+1} = y_n + h\mathrm{f}(t_n, y_n)$ $y_{n+1} = y_n + \frac{h}{2}\left[\mathrm{f}(t_n, y_n) + \mathrm{f}(t_{n+1}, u_{n+1})\right]$ Given: $y_2 = 0.1182861$ (7 sf), we have $u_3 = 0.1182861 + (0.25)(0.4408569) = 0.2285003$ $y_3 = 0.1182861 + \frac{0.25}{2}(0.4408569 + 0.5786247) = 0.2457213$ $u_4 = 0.2457213 + (0.25)(0.5657090) = 0.3871486$ $y_4 = 0.2457213 + \frac{0.25}{2}(0.5657090 + 0.6128514) = 0.3930414$ $= 0.39304 \text{ (5 sf)}$.	This part was generally well done. Students are required to keep their intermediate answers all to 7s.f. as final answer is to be in 5s.f.
(iii) **[4]**	$y_3(t) = y_0 + \int_0^t \mathrm{f}(x, y_2(x))\ \mathrm{d}x$ $= 0 + \int_0^t x - xy_2(x)\ \mathrm{d}x$ $= \int_0^t x - x\left(\frac{x^2}{2} - \frac{x^4}{8}\right) \mathrm{d}x$ $= \int_0^t x - \frac{x^3}{2} + \frac{x^5}{8}\ \mathrm{d}x$ $= \left[\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{48}\right]_0^t = \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{48}$. $y_4(t) = y_0 + \int_0^t \mathrm{f}(x, y_3(x))\ \mathrm{d}x$ $= 0 + \int_0^t x - xy_3(x)\ \mathrm{d}x$ $= \int_0^t x - x\left(\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{48}\right) \mathrm{d}x$ $= \int_0^t x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48}\ \mathrm{d}x$ $= \left[\frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^8}{384}\right]_0^t = \frac{t^2}{2} - \frac{t^4}{8} + \frac{t^6}{48} - \frac{t^8}{384}$. Hence, $y_4(1) = \frac{1}{2} - \frac{1}{8} + \frac{1}{48} - \frac{1}{384} = \frac{151}{384} = 0.39323 \text{ (5 sf)}$.	Most students scored full mark for this part, except some who committed careless mistakes.

(iv) **[2]**	**(i)** gives $y = 0.39347\ (5\text{ sf})$, **(ii)** gives $y_4 = 0.39304\ (5\text{ sf})$, while **(iii)** gives $y_4(1) = 0.39323\ (5\text{ sf})$. Though **(ii)** and **(iii)** give very good approximation to **(i)** when correct to 3 significant figures, **(iii)** seems to give a better approximation to **(i)** than **(ii)** at a higher level of significance. **(ii)** can be improved by reducing the step size. **(iii)** can be improved by having more iterations.	Note that **(iii)** CANNOT be improved by reducing step size as there is no step size for *t* involved. Picard's iteration uses a sequence of functions, instead of numbers, to approximate the solution upon substituting in *t* of interest.

Qn	Suggested Solution	Comment
6(i) **[2]**	Let X be the number of students who clocked at least 10,000 steps daily, out of 300 students. $X \sim \mathrm{B}(300, p)$ Since $n = 300$ is large, $X \sim \mathrm{N}(300p, 300pq)$ approximately where $q = 1 - p$. So $P = \frac{X}{300} \sim \mathrm{N}\left(p, \frac{pq}{300}\right)$ approximately $\hat{p} = \frac{78}{300} = 0.26$ From GC, a 95% confidence interval for p is (0.210, 0.310) (3sf)	Students are reminded of the need to state the distribution. Common mistake: students used $\hat{p}$ in place of p when they stated the distribution.
(ii) **[2]**	Let the number of PE teachers that conducted the survey be k. $\hat{p} = \frac{0.146 + 0.254}{2} = 0.2$ $\hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{30k}} = 0.254 \quad \Rightarrow k = 7$	Mostly well done.
(iii) **[2]**	Note that $\hat{p}$ remains as 0.2 with the inclusion of the missing results. Interval width $= 2 \times 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ Since n is now larger, the interval width becomes smaller.	Most are able to take note that $\hat{p}$ remains unchanged and so the width of interval will become smaller.

Qn	Suggested Solution	Comment
7 **[6]**	H_0 : "Type of car" and "age of car owner" are independent. H_1 : "Type of car" and "age of car owner" are NOT independent. Perform a χ^2 test on independence at the 5% significance level Under H_0, Expected Frequencies (E_{ij}):	Mostly well done for calculating the expected frequencies and the contributions to χ^2_{cal}. However, when it comes to the conclusion, students need to use some suitable levels of significance to provide the appropriate conclusions. For the discussion on the association, students need to touch on the strength of the evidence and how the contributions may provide some evidence as hinted by the question.

Type of car	Age of car owner			Total
	Under 30	30 – 50	Above 50	
Saloon	25.6667	66.3667	39.9666	132
Coupe	7.1944	18.6028	11.2028	37
Hatchback	21.1944	54.8028	33.0028	109
MPV	15.9445	41.2277	24.8278	82
Total	70	181	109	360

Observed Frequencies (O_{ij}):

Type of car	Age of car owner			Total
	Under 30	30 – 50	Above 50	
Saloon	25	67	40	132
Coupe	15	12	10	37
Hatchback	22	55	32	109
MPV	8	47	27	82
Total	70	181	109	360

Contributions to χ^2_{cal} $\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\left(O_{ij}-E_{ij}\right)^2}{E_{ij}}\right)$:

Type of car	Age of car owner		
	Under 30	30 – 50	Above 50
Saloon	0.017316	0.0060439	0.000027801
Coupe	8.4686	2.3436	0.12914
Hatchback	0.030617	0.00070976	0.030469
MPV	3.9584	0.80816	0.19005

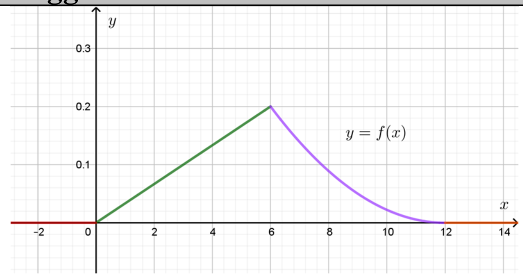
The degrees of freedom, $\upsilon = (4-1)\times(3-1) = 6$

Using the GC, $\chi^2_{\text{cal}} = 15.983$
and $p-\text{value} = 0.01384$

Since $p-\text{value} = 0.01384 < 0.05$, we reject H_0 and conclude that there is sufficient evidence, at the 5% significance level, that there is association between the type of car and the age of car owner.

[2]	However if the test is performed at 1% significance level, H_0 will be barely accepted. Thus, there is some but not strong evidence to suppose that there is association between the type of car and age of car owner. The strongest evidence for association is in the under 30 - coupe cell, where the observed frequencies is much greater than expected under H_0, making the greatest contribution to the χ^2 test statistic.	

Qn	Suggested Solution	Comment
8(i) **[2]**	Firstly, note that for a pair of dice to land on the same score of 1, the probability is $\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$. This is similar for 2, 3 and 4. Hence for a pair of dice to obtain a score of 0 (i.e. no score), the probability is $1-\frac{1}{16}\times 4=\frac{3}{4}$. A score of 2 for the four dice may come from a score of 0 + 2, 1 + 1 or 2 + 0. $\therefore$ Required probability $=\frac{1}{16}\times\frac{3}{4}\times 2+\frac{1}{16}\times\frac{1}{16}=\frac{25}{256}$ (shown).	Some explanations need to be given on how the probability is calculated as it is a "show" question.
(ii) **[3]**	$X \sim \text{Geo}\ (p)$ where $p=\frac{25}{256}$ $\text{Var}\,(X)=\frac{1-p}{p^2}=\frac{59136}{625}$ or 94.6176 (accept 94.6) $P(X=7)=(1-p)^6 p=0.0527$ (3 s.f)	Very well done.
(iii) **[3]**	Assumptions: • The visits to the booth occur randomly and independently. • The average rate of visits to the booth is a constant. • The probability of two or more visitors to the booth occurring within a very short interval is negligible. In practice this may not hold. For example, the decision to play a game might be affected by a friend who has already played the game.	All 4 assumptions ("RIUS") must be stated. For singly, students need to explain what it means in the context.
(iv) **(a)** **[1]**	Let the random variable Y denote the number of visitors at the booth in 10 minutes. $Y \sim \text{Po}\left(\frac{13}{3}\right)$ $P(Y=0)=0.0131$ (3 s.f.)	Take note of the interval being 10 minutes.
(b) **[1]**	$P(Y>3)=1-P(Y\le 3)=0.629$ (3 s.f.)	Common mistake: $P(Y>3)=1-P(Y\le 2)$

Qn	Suggested Solution	Comment
9(i) **[2]**		Students are reminded to extend the graph beyond the end-points and also to give the value of the maximum point
(ii) **[1]**	$P(X \le 6) = \frac{1}{2}(6)(0.2)$ $= 0.6 > 0.5$ Therefore, median demand is less than 6 tonnes.	Most students only stated that the area under the graph from $x = 0$ to $x = 6$ is larger and hence the conclusion, which is not sufficient. The best way is to give the area.
(iii) **[3]**	$P(X \ge 8) = \frac{16}{135}$ $\mathrm{E}(X) = \int_0^6 \frac{1}{30}x^2 \mathrm{d}x + \int_6^8 \frac{1}{180}x(12-x)^2 \mathrm{d}x + 8 \times P(X \ge 8)$ $= \frac{713}{135}$ The expected amount of rice sold each month is 5.28 tonnes (3.s.f.)	Poorly done as students didn't see the need to find $P(X \ge 8) = \frac{16}{135}$ and hence $8 \times P(X \ge 8)$ which contributed to $\mathrm{E}(X)$.
(iv) **[2]**	$P(X \le k) \ge 0.95$ $0.6 + \int_6^k \frac{1}{180}(12-x)^2 \mathrm{d}x \ge 0.95$ Using GC, $k \ge 9$ $\therefore$ The least value of k is 9.	Students are reminded that they should work with inequality so as to conclude the least value of k.
(v) **[2]**	If the storage is extended to 12 tonnes, expected amount of rice sold would be $\mathrm{E}(X) = \int_0^6 \frac{1}{30}x^2 \mathrm{d}x + \int_6^{12} \frac{1}{180}x(12-x)^2 \mathrm{d}x$ $= 5.4$ Therefore, expected increase in profit $= \left(5.4 - \frac{713}{135}\right) \times 480 = 56.8889 \approx 56.89$ The expected increase in monthly profit will be \$56.89.	Answer to this was affected by their attempt in (iii). As such, this was poorly done as well.

<table>
<tr><td>(vi)
[2]</td><td>$\frac{20000}{56.8889} \approx 351.56$ months $\approx$ 29.3 years

Since it would take about 29.3 years for the merchant to break even, it does not seem feasible to extend the storage.</td><td>Most students attempted to look at how long it will take for the merchant to “break even”, but was not given full credit due to mistake in part (v).</td></tr>
</table>

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th>Comment</th></tr>
<tr><td>10
(i)
[5]</td><td>Let X and Y be the LDL levels of hypercholesterolemic Singaporean men on the oat bran cereal diet and corn flakes diet respectively. Let $D = Y - X$ be the difference in LDL levels, and m_D be the population median difference.
<table>
<tr><td>Pair</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td>d</td><td>-0.10</td><td>0.85</td><td>0.45</td><td>0.86</td><td>0.6</td><td>0</td><td>0.16</td><td>0.25</td><td>-0.41</td></tr>
<tr><td>sign</td><td>-</td><td>+</td><td>+</td><td>+</td><td>+</td><td></td><td>+</td><td>+</td><td>-</td></tr>
</table>

Sign Test:
To test $H_0 : m_D = 0$ vs $H_1 : m_D > 0$
Let S_+ be the number of + signs out of 8.
From the table, $s_+ = 6$, $s_- = 2$.
Under H_0, $S_+ \sim B(8, 0.5)$
p-value = $\mathrm{P}(S_+ \geq s_+) = 0.145$ (3.s.f)
Since p-value = $0.145 > 0.05$, we do not reject H_0, and conclude that there is insufficient evidence, at 5% significance, that an oat bran cereal diet is more effective than a corn flakes diet in reducing LDL levels for hypercholesterolemic males.</td><td>Mostly well done for (i) and (ii), except some who needed to improve in their presentation like defining random variables and using median for sign tests.</td></tr>
<tr><td>(ii)
[3]</td><td>Wilcoxon matched-pair signed rank test:
<table>
<tr><td>Pair</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td><td>7</td><td>8</td><td>9</td></tr>
<tr><td>d</td><td>-0.10</td><td>0.85</td><td>0.45</td><td>0.86</td><td>0.6</td><td>0</td><td>0.16</td><td>0.25</td><td>-0.41</td></tr>
<tr><td>Rank</td><td>1</td><td>7</td><td>5</td><td>8</td><td>6</td><td></td><td>2</td><td>3</td><td>4</td></tr>
</table>
To test $H_0 : m_D = 0$ vs $H_1 : m_D > 0$
The sum of the positive ranks, $P = 2+3+5+6+7+8 = 31$ and the sum of the negative ranks, $Q = 1+4 = 5$
So $T = \min(P, Q) = 5$
From table, since $T \leq 5$ we reject H_0 and conclude that there is sufficient evidence at 5% significance, that an oat bran cereal diet is more effective than a corn flakes diet in reducing LDL levels for hypercholesterolemic males.</td><td></td></tr>
</table>

(iii) **[4]**	Assuming that the data are samples from normal distributions, and D follows a normal distribution. For a paired sample t-test, $H_0 : \mu_D = 0$ vs $H_1 : \mu_D > 0$ Under H_0, $T = \dfrac{\overline{D}}{S/\sqrt{n}} \sim t(n-1)$ From the sample, $\overline{d} = 0.29556$, $s^2 = \dfrac{1}{8}\left(2.2908 - \dfrac{2.66^2}{9}\right) = 0.18808$, $n = 9$. Using a paired sample t-test, p-value = 0.0376 Since p-value = 0.0376 < 0.05. Hence we reject H_0 and conclude that there is sufficient evidence at 5% significance level, that an oat bran cereal diet is more effective than a corn flakes diet in reducing LDL levels for hypercholesterolemic males.	Mostly managed to use the right test except a few who opted for 2 sample t-test and it cost them dearly. Students should take note that the question stated “each member of the pair having similar low-density lipoprotein (LDL) levels in January” as justification to use paired sample t-test.
(iv) **[2]**	The conclusion of the two non-parametric tests are different. The Wilcoxon test made use of the magnitude of the differences between matched pairs of the LDL levels, and is hence a better test compared to the sign test as it incorporates more information about the data. Hence the conclusion of the Wilcoxon matched-pair signed rank test is more reliable. The paired sample t-test requires the assumption that the data comes from normal distributions and it depends on whether the parameters of the normal distributions can be found. In this research context it is unlikely for them to be found. Hence using the non-parametric tests would not require the researcher to assume that the data comes from normal distributions.	Most students were able to point out the advantages of using the Wilcoxon test over the sign test, but miss out on the comparison between the paired sample t-test and Wilcoxon test.