1 One of the roots of the equation $w^2 - (6-5\text{i})w + 11 - \text{i} = 0$ is $1+\text{i}$.
Explain why the Conjugate Root Theorem cannot be applied to find the other root. [1]

Determine the other root of the equation $w^2 - (6-5\text{i})w + 11 - \text{i} = 0$ in the form $p + \text{i}q$. [2]

2 **(i)** State a sequence of transformations which transforms the graph of $y = x^2$ to the graph of $y = \frac{1}{a}(x-a)^2 + \frac{1}{a}$, where a is a positive constant. [3]

(ii) A curve has equation $y = \text{f}(x)$, where

$$\text{f}(x) = \begin{cases} x^2 & \text{for } -1 < x \le 1, \\ \frac{1}{2}(x-2)^2 + \frac{1}{2} & \text{for } 1 < x \le 3, \\ 1 & \text{otherwise.} \end{cases}$$

Sketch the curve of $y = \text{f}(x)$ for $-2 \le x \le 5$.
You should state clearly the coordinates of the stationary points. [3]

3 The complex number z is given by $z = 1 + \text{i}a$, where a is a positive real number.

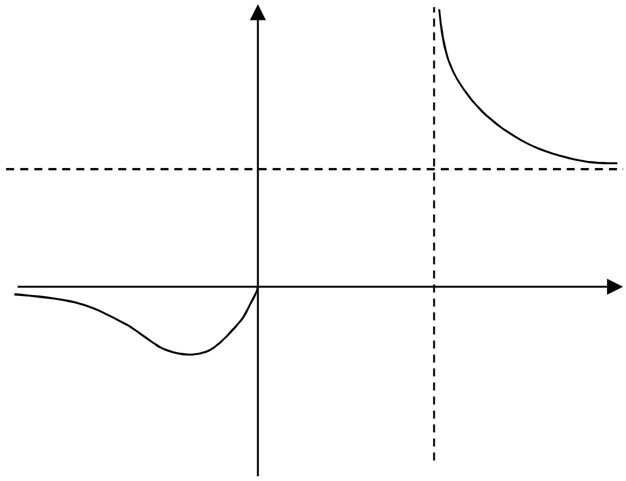
(i) Given that the argument of $(z-3)$ is $\frac{3\pi}{4}$, find the value of a. [2]

The complex number w is given by $w = \text{i}b$, where b is a positive real number. Find

(ii) the modulus of $\frac{w^n}{w^*}$ in terms of b and n, [1]

(iii) the two smallest positive whole number values of n for which $\frac{w^n}{w^*}$ is purely imaginary. [4]

4



The diagram shows the curve $y = \mathrm{f}(x)$. The curve has turning points at $(1,\ b)$ and $(-1, -b)$, and crosses the x-axis at the origin and $(2,\ 0)$. The curve has asymptotes $y = 0$, $y = a$ and $x = 3$. It is given that a and b are positive constants with $a > b$.

On separate diagrams, draw sketches of the graphs of

(a) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(b) $y = \mathrm{f}'(x)$, [3]

(c) $y = |\mathrm{f}(x)|$, [2]

stating, if it is possible to do so, the coordinates of the points where the graphs cross the axes, the coordinates of any turning point(s) and the equations of any asymptotes.

5 It is given that $y^3 = 7 + \cos x$.

(i) Show that $6y\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 3y^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 7 - y^3$. [2]

(ii) By differentiating this result, show that the Maclaurin series for y does not have a term in x^3. [3]

(iii) Hence write down the series expansion of y up to and including the term in x^2. Give the coefficients in exact form. [1]

(iv) Using appropriate expansions from the List of Formulae (MF26), verify the expansion found in part **(iii)**. [3]

6 The sequence $\{u_n\}$ is such that $u_n = \dfrac{ne^{n^2}}{n+1}$ where $n \in \mathbb{Z}^+$. It is given that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.

(i) Find $\displaystyle\sum_{r=1}^{n} \ln(u_r)$ in terms of *n*. [3]

A sequence $\{v_n\}$ is such that $v_n = 2^{\ln(u_n)-n^2}$, $n \in \mathbb{Z}^+$. Another sequence $\{w_n\}$ is defined as $w_n = v_1 \times v_2 \times v_3 \times \cdots \times v_n$, $n \in \mathbb{Z}^+$.

(ii) Hence, or otherwise, find w_n in terms of *n* and determine, with a reason, if $\{w_n\}$ is convergent. [5]

7 **(a)** **(i)** Find $\int \cos 3x \cos x \, \mathrm{d}x$. [3]

(ii) Hence, find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} x \cos 3x \cos x \, \mathrm{d}x$. [4]

(b) Find, in terms of *a*, $\displaystyle\int_{-a}^{a} \left| x(x-2a) \right| \mathrm{d}x$, where *a* is a positive constant. [3]

8 Eden is a new engineer for the Small Rail Transit (SRT) train. His current project is to devise a new schedule for the train to increase its efficiency.
In the current schedule, the train will run for 5 days in one cycle. For the first day of the first cycle, the train runs for 100 km. On the 2nd and on the 3rd day of each cycle, the train will run 30 km more than the first day of that cycle. On the 4th and on the 5th day of each cycle, the train will run 70 km more compared to the first day of that cycle. On the first day of the k^{th} cycle, the train will run 30 km more compared to the last day of the $(k-1)^{\text{th}}$ cycle.

(i) It was found out that the train broke down after travelling 10 000 km. By forming an inequality, state the cycle and the day of the cycle that the train was in during the breakdown. [6]

Eden decides to modify the schedule such that the train will now operate every week from Monday to Saturday. On the first day of operations, a Monday, the train runs for 100 km. On every Tuesday, Thursday and Saturday, the distance travelled will be increased by 20% compared to the previous operating day. However, on every Monday, Wednesday and Friday, there will be a 8% decrease in the distance travelled compared to the previous operating day.

(ii) Find the total distance travelled by the train in 2*n* days of operations. [3]

(iii) Find the minimum number of days that the train has run, such that Eden would have improved the previous schedule (i.e. the total distance run before breakdown is more than 10 000 km). [2]

9 The two lines l_1 and l_2 have equations $x = \dfrac{z}{-2} - 2,\ y = 0$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix}$ respectively, where $\mu \in \mathbb{R}$ and q is a constant. The point A has coordinates $(4,\ 3,\ k)$ where k is a constant and the two planes p_1 and p_2 have equations $4x = -8 + y - 2z$ and $y + tz = 2t$ respectively, where t is a constant.

(i) Find a vector equation of l_1 and hence show that l_1 lies in p_1. [3]

(ii) Find the coordinates of the foot of perpendicular from A to p_1. Express your answer in terms of k. [3]

(iii) Given that the angle between l_1 and l_2 is 60°, find the possible values of q. [2]

(iv) Given that $q = 1$ and that the point $B(1,\ 2,\ 2)$ is equidistant from l_2 and p_2, find the possible values of t. [6]

10 **(a)** A skier slides freely from rest at a particular slope at a ski resort and t seconds later, he is sliding down at a speed of v ms^{-1}. He experienced gravitational force, friction and air resistance which affect v. The rate of change of v is modelled as $\dfrac{\mathrm{d}v}{\mathrm{d}t} = 7.2 - kv^2$. It is given that when his speed reaches 30 ms^{-1}, the rate of change of his speed is 5.4 ms^{-2}.

By solving the differential equation, show that $v = \dfrac{60\,(\mathrm{e}^{mt} - 1)}{\mathrm{e}^{mt} + 1}$, where m is a positive constant to be found. [6]

(b) The owner of the ski resort bought new sets of ski boards. He proposes that the velocity (y ms^{-1}) of a skier on the slope now satisfies this differential equation where x is the distance travelled from a starting point (x metres).

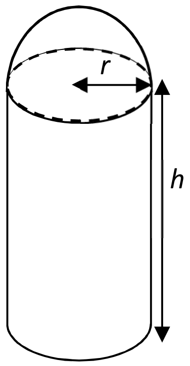
$$3x\frac{\mathrm{d}y}{\mathrm{d}x} = 3y - 4 + x, \text{ where } x > 1$$

Using the substitution $y = ux$, show that the differential equation can be reduced to $3x^2 \dfrac{\mathrm{d}u}{\mathrm{d}x} = x - 4$ and hence find the general solution for y in terms of x. [5]

Suggest a suitable reason why the differential equation would not be appropriate in modelling how the velocity varies for large distances. [1]

11 A company designs a candy container in the shape shown below. The container is made up of an open cylinder of height h cm and radius r cm, with a hollow hemispherical lid of radius r cm. Its lid costs a cents per cm^2, where a is a fixed constant and the cylinder costs half as much per cm^2 to manufacture. The company wants to design the container such that its total volume is 300 cm^3. Find, using differentiation, the exact value of r and h that will minimise the cost. [7]

[Volume of sphere, $V = \frac{4}{3}\pi r^3$; Surface area of sphere, $S = 4\pi r^2$]



The annual profit made by the company is P (in ten thousand dollars per year). The economist believes that the relationship between P and t (in years) is given by

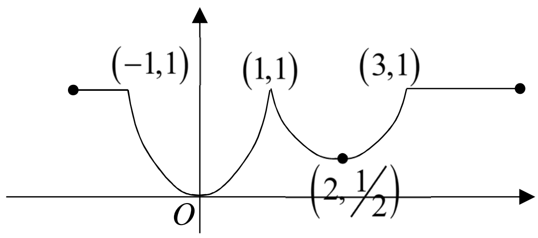
$$P = \frac{\sqrt{\ln(t+1)}}{t+1}, \text{ for } 0 \le t \le 5.$$

Sketch the graph of P against t, stating the coordinates of the turning point and the end points. [2]

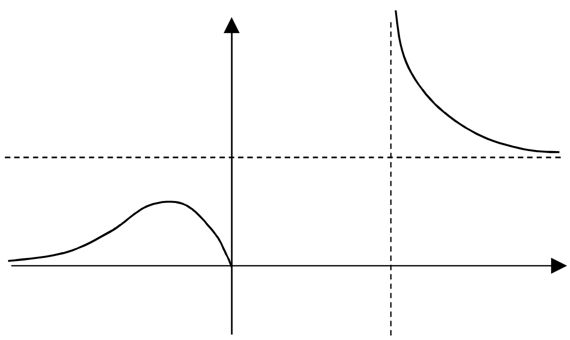
Find, by integration, the exact area of the region bounded by the curve, the t-axis and the line t =5. Give an interpretation of the area that you have found, in the context of the question. [3]

− End Of Paper −

Yishun Junior College
2018 JC2 H2 Math Preliminary Examination
Paper 1 Solutions

Qns	Solutions
1	The coefficients of the polynomial (quadratic) equation are not real. Hence the Conjugate Root Theorem cannot be applied. $w^2 - (6-5\mathrm{i})w + 11 - \mathrm{i} = 0$ $w = \dfrac{(6-5\mathrm{i}) \pm \sqrt{(6-5\mathrm{i})^2 - 4(1)(11-\mathrm{i})}}{2}$ $= \dfrac{(6-5\mathrm{i}) \pm (4-7\mathrm{i})}{2}$ $= 5-6\mathrm{i}$ or $1+\mathrm{i}$ Hence the other root is $5-6\mathrm{i}$.
2	(i) 1. A translation of a units in the direction of the x-axis. 2. A scaling parallel to the y-axis by a factor of $\frac{1}{a}$. 3. A translation of $\frac{1}{a}$ units in the direction of the y-axis. (ii) 

3	(i) $z-3=-2+\mathrm{i}a$ $\arg(z-3)=\pi-\tan^{-1}\left(\frac{a}{2}\right)$ $\pi-\tan^{-1}\left(\frac{a}{2}\right)=\frac{3\pi}{4}$ $\tan^{-1}\left(\frac{a}{2}\right)=\frac{\pi}{4}$ $\frac{a}{2}=1$ $a=2$ (ii) $\left\|\frac{w^n}{w^*}\right\|=\frac{\|w\|^n}{\|w^*\|}=\frac{\|w\|^n}{\|w\|}=\frac{b^n}{b}=b^{n-1}$ (iii) $\arg\left(\frac{w^n}{w^*}\right)=\arg(w^n)-\arg(w^*)$ $=n\arg(w)-(-\arg(w))$ $=n\arg(w)+\arg(w)$ $=n\left(\frac{\pi}{2}\right)+\left(\frac{\pi}{2}\right)$ $=\left(\frac{\pi}{2}\right)(n+1)$ Since $\frac{w^n}{w^*}$ is purely imaginary. $\arg\left(\frac{w^n}{w^*}\right)=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},...$ $\left(\frac{\pi}{2}\right)(n+1)=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},...$ $n+1=1,3,5,...$ $\therefore$ two smallest positive whole number $n=2,4$

4	(a) $y = \frac{1}{\mathrm{f}(x)}$, y (b) $y = \mathrm{f}'(x)$ y (c) $y = \lvert \mathrm{f}(x) \rvert$ 

5	(i) $y^3 = 7+\cos x$ $3y^2\dfrac{dy}{dx} = -\sin x$ $6y\left(\dfrac{dy}{dx}\right)^2 + 3y^2\dfrac{d^2y}{dx^2} = -\cos x$ $6y\left(\dfrac{dy}{dx}\right)^2 + 3y^2\dfrac{d^2y}{dx^2} = -\left(y^3-7\right)$ $6y\left(\dfrac{dy}{dx}\right)^2 + 3y^2\dfrac{d^2y}{dx^2} = 7-y^3$ (ii) When $x=0$, $y=2$, $\dfrac{dy}{dx}=0$, $\dfrac{d^2y}{dx^2}=-\dfrac{1}{12}$ $6\left(\dfrac{dy}{dx}\right)^3 + 12y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) +$ $6y\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right) + 3y^2\left(\dfrac{d^3y}{dx^3}\right) = -3y^2\dfrac{dy}{dx}$ When $x=0$, $\dfrac{d^3y}{dx^3}=0$ Since $\dfrac{d^3y}{dx^3}=0$, term in $x^3 = \dfrac{f'''(0)}{3!}x^3 = 0$. Hence Maclaurin series of y does not have a term in x^3. (iii) $y = 2+\left(-\dfrac{1}{12}\right)\dfrac{x^2}{2!}+...$ $= 2-\dfrac{1}{24}x^2+...$

	(iv) $y^3 = 7 + \cos x$ $y = \left[7 + \left(1 - \frac{x^2}{2} + \ldots\right)\right]^{\frac{1}{3}}$ $y = \left(8 - \frac{x^2}{2} + \ldots\right)^{\frac{1}{3}}$ $= 2\left(1 - \frac{x^2}{16} + \ldots\right)^{\frac{1}{3}}$ $= 2\left[1 + \frac{1}{3}\left(-\frac{x^2}{16}\right) + \ldots\right]$ $= 2 - \frac{1}{24}x^2 + \ldots$
6	(i) $\sum_{r=1}^{n} \ln\left(\frac{re^{r^2}}{r+1}\right) = \sum_{r=1}^{n}\left[\ln r + \ln e^{r^2} - \ln(r+1)\right]$ $= \sum_{r=1}^{n}\left[\left(\ln r - \ln(r+1)\right) + r^2\right]$ $= \sum_{r=1}^{n} \ln r - \ln(r+1) + \sum_{r=1}^{n} r^2$ $\begin{array}{lll} & \ln 1 & \cancel{-\ln 2} \\ = + & \cancel{\ln 2} & \cancel{-\ln 3} + \frac{n}{6}(n+1)(2n+1) \\ + & \cancel{\ln 3} & \cancel{-\ln 4} \\ & \vdots & \\ + & \cancel{\ln(n-1)} & \cancel{-\ln n} \\ + & \cancel{\ln n} & -\ln(n+1) \end{array} + \frac{n}{6}(n+1)(2n+1)$ $= \frac{n}{6}(n+1)(2n+1) - \ln(n+1)$ (ii) $v_1 = 2^{\ln(u_1) - 1^2} = \frac{2^{\ln(u_1)}}{2^{1^2}}$ $v_2 = 2^{\ln(u_2) - 2^2} = \frac{2^{\ln(u_2)}}{2^{2^2}}$ …

	$v_n = 2^{\ln(u_n)-n^2} = \dfrac{2^{\ln(u_n)}}{2^{n^2}}$ $v_1 \times v_2 \times \cdots \times v_n = \dfrac{2^{\ln(u_1)} \times 2^{\ln(u_2)} \times \cdots \times 2^{\ln(u_n)}}{2^1 \times 2^4 \times \cdots \times 2^{n^2}}$ $= \dfrac{2^{\sum_{r=1}^{n}\ln(u_r)}}{2^{\sum_{r=1}^{n} r^2}}$ $= \dfrac{2^{\frac{n}{6}(n+1)(2n+1)-\ln(n+1)}}{2^{\frac{n}{6}(n+1)(2n+1)}}$ $= 2^{-\ln(n+1)}$ $v_1 \times v_2 \times \cdots \times v_n = \dfrac{1}{2^{\ln(n+1)}}$ As $n \to \infty, \dfrac{1}{2^{\ln(n+1)}} \to 0$, therefore convergent.
7	(ai) $\int \cos 3x \cos x \, \mathrm{d}x = \frac{1}{2}\int 2\cos 3x \cos x \, \mathrm{d}x$ $= \frac{1}{2}\int (\cos 4x + \cos 2x) \, \mathrm{d}x$ $= \frac{1}{2}\left(\frac{1}{4}\sin 4x + \frac{1}{2}\sin 2x\right) + c$ $= \frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + c$ (aii) $\int_0^{\frac{\pi}{4}} x\cos 3x \cos x \, \mathrm{d}x = x\left[\frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\left(\frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x\right) \mathrm{d}x$ $= \left(\frac{\pi}{4}\right)\left[\frac{1}{8}\sin 4\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right] - \left[-\frac{1}{32}\cos 4x - \frac{1}{8}\cos 2x\right]_0^{\frac{\pi}{4}}$ $= \left(\frac{\pi}{4}\right)\left[\frac{1}{8}\sin \pi + \frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right] +$ $\left[\frac{1}{32}\cos 4\left(\frac{\pi}{4}\right) + \frac{1}{8}\cos 2\left(\frac{\pi}{4}\right) - \left(\frac{1}{32} + \frac{1}{8}\right)\right]$ $= \frac{\pi}{16} + \left[-\frac{1}{32} - \frac{1}{32} - \frac{1}{8}\right] = \frac{\pi - 3}{16}$

	(b) $\int_{-a}^{a} \left\| x(x-2a) \right\| \ \mathrm{d}x$ $=\int_{-a}^{0} x(x-2a)\ \mathrm{d}x+\int_{0}^{a} -x(x-2a)\ \mathrm{d}x$ $=\left[\frac{x^3}{3}-ax^2\right]_{-a}^{0}-\left[\frac{x^3}{3}-ax^2\right]_{0}^{a}$ $=2a^3$
8	(i) $S_n \geq 10000$ $\frac{n}{2}[2(700)+(n-1)(500)] \geq 10000$ $\frac{n}{2}(900+500n) \geq 10000$ $500n^2+900n-20000 \geq 0$ From GC, $n \leq -7.2883$ or $n \geq 5.4883$ (rejected) The train has completed 5 cycles $S_5=\frac{5}{2}[2(700)+4(500)]$ $=8500$ The train broke down after 1500km in the 6th cycle The train will run 600 km on the first day of the 6th cycle. The train will break down on the 3rd day of the 6th cycle. (ii) Let O_n be the sum of distance the train runs on *n* odd numbered days. $O_n=100+0.92(1.2)(100)+0.92^2(1.2)^2(100)$ $+\ldots+0.92^{n-1}(1.2)^{n-1}(100)$ $=\frac{100(1.104^n-1)}{1.104-1}$ $=961.54(1.104^n-1)$ Let E_n be the sum of distance the train runs on *n* even numbered days $E_n=(1.2)100+0.92(1.2)^2(100)+0.92^2(1.2)^3(100)$ $+\ldots+0.92^{n-1}(1.2)^n(100)$ $=\frac{120(1.104^n-1)}{1.104-1}$ $=1153.8(1.104^n-1)$

	Let S_n be the total distance covered by n days. $S_{2n} = O_n + E_n$ $= 961.54\left(1.104^n - 1\right) + 1153.8\left(1.104^n - 1\right)$ $= 2115.4\left(1.104^n - 1\right)$ (iii) $S_{2n} > 10000$ $2115.4\left(1.104^n - 1\right) > 10000$ $1.104^n > 5.7275$ $n > 17.640$ Therefore, the train must complete at least <u>36</u> days of travelling.
9	(i) $l_1: \ x+2 = \frac{z}{-2}$ Let $\lambda = x+2 = \frac{z}{-2}$ $x = -2+\lambda$ $y = 0$ $z = -2\lambda$ $\boldsymbol{r} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$ $\left[\begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right] \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ $= -8 + 4\lambda - 4\lambda$ $= -8$ for all values of λ. Hence all points on l_1 lies in p_1. Hence, l_1 lies in p_1. (ii) Let the foot of perpendicular from A to p_1 be N. Equation of the line that passes through A and perpendicular to p_1 is $l_{AN}: \boldsymbol{r} = \begin{pmatrix} 4 \\ 3 \\ k \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}, \quad \alpha \in \mathbb{R}$.

Since N lies on l_{AN}, $\overrightarrow{ON} = \begin{pmatrix} 4+4\alpha \\ 3-\alpha \\ k+2\alpha \end{pmatrix}$, for some $\alpha \in \mathbb{R}$

$$\begin{pmatrix} 4+4\alpha \\ 3-\alpha \\ k+2\alpha \end{pmatrix} \bullet \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = -8$$

$$16+16\alpha-3+\alpha+2k+4\alpha=-8$$

$$21\alpha=-21-2k$$

$$\alpha=-1-\frac{2k}{21}$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 4-4-\frac{8k}{21} \\ 3+1+\frac{2k}{21} \\ k-2-\frac{4k}{21} \end{pmatrix} = \begin{pmatrix} -\frac{8k}{21} \\ 4+\frac{2k}{21} \\ -2+\frac{17k}{21} \end{pmatrix}$$

Hence, N is the point $\left(-\frac{8}{21}k, 4+\frac{2}{21}k, \frac{17}{21}k-2\right)$

(iii)

Direction vector of l_2 is $\begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix}$

$$\cos 60^o = \frac{\left|\begin{pmatrix} 0 \\ q \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}\right|}{\sqrt{q^2+4}\sqrt{5}}$$

$$\frac{1}{2} = \frac{4}{\sqrt{5q^2+20}}$$

$$\sqrt{5q^2+20}=8$$

$$q^2=\frac{44}{5}$$

$$q=\pm\sqrt{\frac{44}{5}} = \pm 2.97 \textbf{ (3 s.f.)}$$

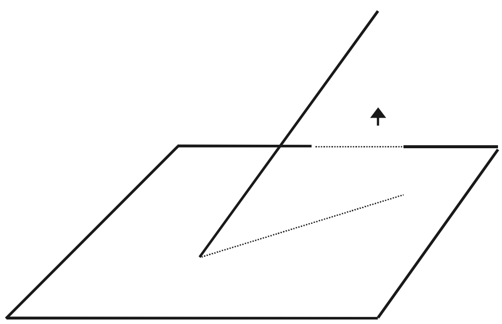
(iv)

$B(1,2,2)$

$$\overrightarrow{CB}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

Shortest distance from B to the line l_2:

$$=\frac{\left|\begin{pmatrix}0\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}\right|}{\sqrt{1+2^2}}=\frac{\left|\begin{pmatrix}4\\0\\0\end{pmatrix}\right|}{\sqrt{1+2^2}}=\frac{4}{\sqrt{5}}$$



Let D be any point on plane p_2.

$$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix} \qquad \overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

	Shortest distance from B to plane p_2 $=\frac{\left\lvert\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}0\\1\\t\end{pmatrix}\right\rvert}{\sqrt{1+t^2}}=\frac{2}{\sqrt{1+t^2}}$ Since point $B(1,2,2)$ is equidistant from l_2 and p_2, $\frac{4}{\sqrt{5}}=\frac{2}{\sqrt{1+t^2}}$ $\sqrt{1+t^2}=\frac{\sqrt{5}}{2}$ $t^2=\frac{1}{4}$ $t=\pm\frac{1}{2}$
10	(a) $\frac{\mathrm{d}v}{\mathrm{d}t}=7.2-kv^2$ When $v=30, \frac{\mathrm{d}v}{\mathrm{d}t}=5.4$ $5.4=7.2-k\,(30^2)$ $k=0.002$ $\frac{\mathrm{d}v}{\mathrm{d}t}=7.2-0.002v^2$ $\int\frac{1}{7.2-0.002\,v^2}\,\mathrm{d}v=\int 1\,\mathrm{d}t$ $\frac{1}{0.002}\int\frac{1}{3600-v^2}\,\mathrm{d}v=\int 1\,\mathrm{d}t$ $\frac{500}{2(60)}\ln\left\lvert\frac{60+v}{60-v}\right\rvert=t+C$ $\frac{25}{6}\ln\left\lvert\frac{60+v}{60-v}\right\rvert=t+C$ $\ln\left\lvert\frac{60+v}{60-v}\right\rvert=\frac{6}{25}t+\frac{6}{25}C$ $\frac{60+v}{60-v}=\pm e^{\frac{6}{25}t+\frac{6}{25}C}$ $\frac{60+v}{60-v}=Ae^{0.24t},\ A=\pm e^{0.24C}$

When $t=0, v=0$

$\therefore A=1$

$\therefore \frac{60+v}{60-v}=e^{0.24t}$

$60+v=(60-v)\,e^{0.24t}$

$60+v=60\,e^{0.24t}-ve^{0.24t}$

$v(1+e^{0.24t})=60\,e^{0.24t}-60$

$v=\frac{60\,(e^{0.24t}-1)}{(e^{0.24t}+1)}$

(b)

$3x\frac{\mathrm{d}y}{\mathrm{d}x}=3y-4+x$

$y=ux$

$\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}u}{\mathrm{d}x}+u$

Sub eq (2) into eq (1)

$3x(x\frac{\mathrm{d}u}{\mathrm{d}x}+u)=3ux-4+x$

$3x^2\frac{\mathrm{d}u}{\mathrm{d}x}=x-4$ (shown)

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{x-4}{3x^2}$

$u=\int\left(\frac{1}{3x}-\frac{4}{3x^2}\right)\mathrm{d}x$

$\quad=\frac{1}{3}\ln|x|+\frac{4}{3x}+C$

$\frac{y}{x}=\frac{1}{3}\ln x+\frac{4}{3x}+C$

$y=\frac{1}{3}x\ln x+\frac{4}{3}+Cx$

Reason:

(1) According to the model, for positive values of C, the velocity would become very large/increase infinitely/increase to infinity .

(2) For certain negative values of the arbitrary constant, the velocity would decrease to negative infinity.

11	$\pi r^2 h+\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)=300$ $\pi r^2\left(h+\frac{2}{3}r\right)=300$ $h+\frac{2}{3}r=\frac{300}{\pi r^2}$ $h=\frac{300}{\pi r^2}-\frac{2}{3}r$ $C=\left(\pi r^2+2\pi rh\right)\left(\frac{a}{2}\right)+\frac{1}{2}\left(4\pi r^2\right)a$ $=\frac{5}{2}a\pi r^2+a\pi rh$ $=\frac{5}{2}a\pi r^2+a\pi r\left(\frac{300}{\pi r^2}-\frac{2}{3}r\right)$ $=\frac{5}{2}a\pi r^2+\frac{300a}{r}-\frac{2a\pi r^2}{3}$ $=\frac{11}{6}a\pi r^2+\frac{300a}{r}$ $\frac{\mathrm{d}C}{\mathrm{d}r}=\frac{11}{3}a\pi r-\frac{300a}{r^2}$ For minimum cost, $\frac{\mathrm{d}C}{\mathrm{d}r}=0$ $\frac{11}{3}a\pi r-\frac{300a}{r^2}=0$ $\frac{11}{3}a\pi r^3=300a$ $r^3=\frac{900}{11\pi}$ $r=\sqrt[3]{\frac{900}{11\pi}}$

$$h=\frac{300}{\pi\left(\sqrt[3]{\frac{900}{11\pi}}\right)^2}-\frac{2}{3}\left(\sqrt[3]{\frac{900}{11\pi}}\right)$$

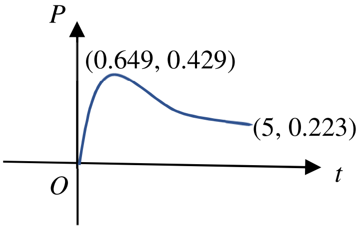
$$=\sqrt[3]{\frac{900}{11\pi}}\left[\frac{300}{\pi}\left(\frac{900}{11\pi}\right)^{-1}-\frac{2}{3}\right]$$

$$=\sqrt[3]{\frac{900}{11\pi}}\left[\frac{300}{\pi}\left(\frac{11\pi}{900}\right)-\frac{2}{3}\right]$$

$$=3\sqrt[3]{\frac{900}{11\pi}}$$

$$\frac{\mathrm{d}^2C}{\mathrm{d}r^2}=\frac{11}{3}a\pi+\frac{600a}{r^3}>0 \text{ as } a,\, r>0$$

Thus, cost is a minimum when $r=\sqrt[3]{\frac{900}{11\pi}}$ and $h=3\sqrt[3]{\frac{900}{11\pi}}$



$$\text{Area}=\int_0^5 \frac{\sqrt{\ln(t+1)}}{t+1}\mathrm{d}t$$

$$=\left[\frac{\left(\ln\left(t+1\right)\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^5$$

$$=\frac{2}{3}[\ln 6]^{\frac{3}{2}}$$

The area represents the total profit made in the five years.

Section A: Pure Mathematics [40 marks]

1 Without using a calculator, solve the inequality $\dfrac{x^2-2x+1}{-2+5x-2x^2} \leq 0$. [3]

Hence solve $\dfrac{x^2-2|x|+1}{-2+5|x|-2x^2} \leq 0$. [2]

2 The complex numbers z and w satisfy the simultaneous equations

$$\mathrm{i}w+z=1 \quad \text{and} \quad zw^*=2-6\mathrm{i}.$$

Find z and w. [5]

3 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 2-\frac{1}{(x-1)(x-2)}, \quad x\in\mathbb{R},\ x\neq 1,\ x\neq 2,$$

$$\mathrm{g}: x \mapsto \sin x, \quad x\in\mathbb{R},\ -\frac{\pi}{2}<x<\frac{\pi}{2}.$$

(i) Explain why the composite function fg exists. Find fg(x) and state the domain and range of fg. [4]

(ii) Sketch the graph of f and explain why the function f^{-1} does not exist. [2]

(iii) The domain of f is now further restricted to $(-\infty, a)$. Find the largest value of a such that the function f^{-1} exists. [1]

(iv) Using your answer to part **(iii)**, define f^{-1} in a similar form. [3]

4 Relative to the origin O, the position vectors of points A, B and N are **a**, **b** and **n** respectively. It is given that N lies on AB, between A and B, such that $\mathbf{a}\cdot\mathbf{n}=\mathbf{b}\cdot\mathbf{n}$ and $\angle AON$ and $\angle BON$ are $30°$ and $60°$ respectively.

(i) Show that AB is perpendicular to ON. [2]

(ii) State the geometrical meaning of $|\mathbf{a}\times\mathbf{n}|$. [1]

(iii) State the geometrical meaning of $\left|\mathbf{a}\times\dfrac{\mathbf{n}}{|\mathbf{n}|}\right|$ and show that its value is $k|\mathbf{n}|$, where k is an exact value to be found. [2]

(iv) Find the ratio $AN:NB$ and express **n** in terms of **a** and **b**. [3]

5 **(a)** The displacement of a particle, t seconds after leaving a fixed point O, is modelled by the curve C with parametric equations $x = \mathrm{e}^t - 1, y = t - \mathrm{e}^t$.

(i) The normal to the curve at the point $P\left(\mathrm{e}^p - 1,\ p - \mathrm{e}^p\right)$ meets the y-axis at the point Q. Find the coordinates of Q. [4]

(ii) Find the rate of change of the particle's y-coordinate when $y = -5$. [2]

(b) The region R is bounded by the curve with equation $y = \dfrac{x}{1+x^2}$, the line $x = \sqrt{3}$ and the x-axis. Using the substitution $x = \tan\theta$, find the exact volume generated when R is rotated completely about the x-axis. [6]

Section B: Probability and Statistics [60 marks]

6 A bag contains 3 red balls and 2 white balls, which are indistinguishable except for their colours. Two balls are drawn at random and without replacement from the bag. If the balls are of different colours, two fair coins are tossed and the number of heads is recorded; if the balls are of the same colour, a fair coin is tossed and the number of heads is recorded. The random variable X is the number of heads recorded.

(i) Show that $\mathrm{P}(X = 0) = \dfrac{7}{20}$. [2]

(ii) Find the probability distribution of X. [2]

(iii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

(iv) Find the probability that the balls are of different colours given that no heads are obtained. [2]

(v) Find the probability that either no heads are obtained or the balls obtained are of the same colour or both. [2]

7 Ten cards each has a single letter or a digit written on them. The letters are *A, B, C, D, E* and the digits are *1, 2, 3, 4, 5*.

(i) Find the number of ways in which the 10 cards can be divided into two sets with each set consisting of 5 cards. [1]

Find the number of ways of arranging the ten cards in a row such that

(ii) all the consonants are next to each other and all the digits are next to each other, [2]

(iii) the letters must be in alphabetical order from left to right. [2]

The ten cards are now arranged at random in a circle.

(iv) Find the probability that no two odd digits are next to each other. [3]

8 A manufacturer produces a large number of coloured glowing sticks, which are packed in packets of 8. Each packet consists of randomly chosen coloured glowing sticks. On average, 12% of the glowing sticks are green.

(i) Explain the significance of 'large number' in the first sentence. [1]

(ii) Find the probability that a randomly chosen packet of glowing sticks contains at least 3 green glowing sticks. [2]

(iii) Ian buys 40 randomly chosen packets of glowing sticks for a concert. Find the probability that fewer than 6 of these packets contain at least 3 green glowing sticks. [2]

On average, the proportion of glowing sticks that are blue is p. It is known that the modal number of blue glowing sticks in a packet is 3.

(iv) Use this information to find exactly the range of values that p can take. [4]

9 A new Internet service provider in the market, Y-Fai, decides to investigate the effect of the distance from its router on its wifi signal. The most convenient way to express wifi signal strength is by using dBm, which stands for decibels relative to a milliwatt. The signal strength measured in dBm is negative in value and a value closer to 0 signifies a stronger wifi signal. An employee measures the signal strength (y dBm) at various fixed distances away from its router (x m) as follows.

x	1	1.5	2	2.5	3	10	15
y	–30	–60	–100	–76	–79	–83	–88

(i) Draw a scatter diagram for these values. On your diagram, circle the data point that seems to be unexpected and suggest a possible reason for the anomaly. [2]

For parts **(ii)** and **(iii)** of this question, you should **exclude** the anomaly.

(ii) Explain from your scatter diagram why the relationship between x and y should not be modelled by an equation of the form $y = ax + b$. [1]

(iii) Which of the formulae $y = \dfrac{c}{x} + d$ and $y = \dfrac{e}{x^2} + f$, where c, d, e and f are constants, is the better model for the relationship between y and x? Explain fully how you decided, and find the constants for the better formula. [4]

(iv) Use the formula you chose from part **(iii)** to estimate the signal strength when the distance away from its router is 5 m. Explain why you would expect this estimate to be reliable. [2]

10 At an early stage in analysing the marks scored by a large number of candidates in an examination paper, the Examination Board claims that the mean mark of candidates is 49. To test this claim, a random sample of 100 candidates is taken and the marks, x, are summarised as follows:

$$\sum(x-49)=313 \text{ and } \sum(x-49)^2 = 23\ 280.$$

(i) Suggest a reason why, in this context, the given data is summarised in terms of $(x-49)$ rather than x. [1]

(ii) Calculate unbiased estimates of the population mean and variance. [2]

(iii) Test, at the 5% significance level, whether the Examination Board's claim is valid. [4]

(iv) State, giving a valid reason, whether the Examination Board needs to make any assumptions about the population in order for the test to be valid. [1]

It is subsequently found that the population standard deviation for the paper is 11. The Examination Board now decides to test the claim that the mean mark of the candidates has increased. A new random sample of 50 candidates is chosen and the mean is k. It is assumed that the marks of the candidates are normally distributed. A test at the 4% significance level indicates that the mean mark of candidates has not increased.

(v) Find the set of possible values of k. [4]

11 The lifespan of light bulbs manufactured by OceanDoor is normally distributed with a mean of 800 hours and a standard deviation of 70 hours. The lifespan of light bulbs manufactured by DaRice is also normally distributed with a standard deviation of 80 hours. It is known that there is a 50% chance that the lifespan of DaRice light bulbs is less than 700 hours.

(i) A random sample of six DaRice light bulbs is chosen for quality check. Find the probability that the mean lifespan of the six DaRice light bulbs differs from half the lifespan of a OceanDoor light bulb by at least 200 hours. [4]

(ii) Find the least value of n such that the probability that the average lifespan of n OceanDoor light bulbs is at most 825 hours is at least 0.99. [2]

(iii) Find the probability that the total lifespan of a randomly selected OceanDoor light bulb and a randomly selected DaRice light bulb is more than 1450 hours. [2]

(iv) Find the probability that a randomly selected OceanDoor light bulb and a randomly selected DaRice light bulb each has lifespan more than 725 hours. [2]

(v) Explain why the value in **(iii)** is larger than **(iv)**. [1]

(vi) State an assumption needed for all your above calculations to be valid [1]

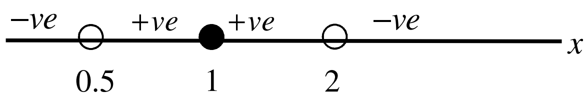
− End Of Paper −

Yishun Junior College
2018 JC2 H2 Math Preliminary Examination
Paper 2 Solutions

<table>
<tr><th>Qns</th><th>Solutions</th></tr>
<tr><td>1</td><td>

$$\frac{x^2-2x+1}{-2+5x-2x^2} \le 0$$

$$\frac{(x-1)^2}{(2x-1)(2-x)} \le 0$$



Therefore, $x<0.5$ or $x>2$ or $x=1$

Hence $|x|<0.5$ or $|x|>2$ or $|x|=1$

$-0.5<x<0.5$ or $x>2$ or $x<-2$ or $x=1$ or -1

</td></tr>
<tr><td>2</td><td>

$\mathrm{i}w+z=1$

$z=1-\mathrm{i}w$ -- Eqn 1

$zw^*=2-6\mathrm{i}$ -- Eqn 2

Subst. Eqn 1 into Eqn 2,

$(1-\mathrm{i}w)w^*=2-6\mathrm{i}$

$w^*-\mathrm{i}ww^*=2-6\mathrm{i}$

Let $w=a+b\mathrm{i}$,

$(a-b\mathrm{i})-\mathrm{i}(a+b\mathrm{i})(a-b\mathrm{i})=2-6\mathrm{i}$

$a-b\mathrm{i}-\mathrm{i}(a^2+b^2)=2-6\mathrm{i}$

$a+\mathrm{i}(-b-a^2-b^2)=2-6\mathrm{i}$

Comparing,

$a=2$,

$-b-2^2-b^2=-6$

$b^2+b-2=0$

$(b-1)(b+2)=0$

$b=1$ or $b=-2$

Therefore $w=2+\mathrm{i}$ or $2-2\mathrm{i}$

When $w=2+\mathrm{i}$, $z=1-\mathrm{i}(2+\mathrm{i})=2-2\mathrm{i}$

When $w=2-2\mathrm{i}$, $z=1-\mathrm{i}(2-2\mathrm{i})=-1-2\mathrm{i}$

</td></tr>
</table>

3

(i) $\mathrm{Rg}=(-1,1)$, $\mathrm{D_f}=(-\infty,\infty)\setminus\{1,2\}$

$\mathrm{Rg}\subseteq \mathrm{D_f}$

$\therefore$ fg exists

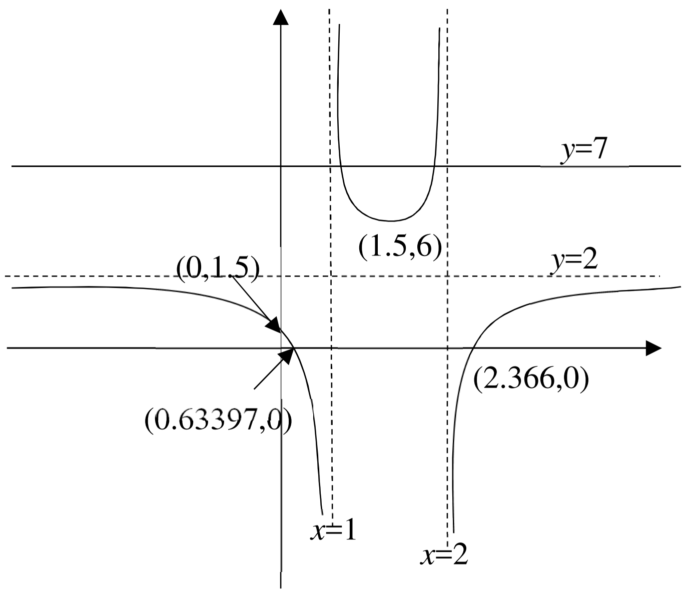
$\mathrm{fg}(x)=\mathrm{f}(\sin x)$

$$=2-\frac{1}{(\sin x-1)(\sin x-2)}$$

$\mathrm{D_{fg}}=\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\mathrm{R_{fg}}=\left(-\infty,\frac{11}{6}\right)$

(ii) $\mathrm{f}(x)=2-\frac{1}{(x-1)(x-2)}$



Since the horizontal line, $y=7$ cuts the graph more than once, f is not one-one, $\therefore$ f^{-1} does not exist.

(iii)

Largest value of a $=\frac{3}{2}$

(iv) Let $y=\mathrm{f}\left(x\right)$

$$y=2-\frac{1}{x^2-3x+2}$$

$$\frac{1}{x^2-3x+2}=2-y$$

$$x^2-3x+2=\frac{1}{2-y}$$

$$x^2-3x+2-\frac{1}{2-y}=0$$

$$x=\frac{-\left(-3\right)\pm\sqrt{9-4\left(2-\frac{1}{2-y}\right)}}{2}$$

$$=\frac{3}{2}\pm\frac{1}{2}\sqrt{1+\frac{4}{2-y}}$$

Reject $x=\frac{3}{2}+\frac{1}{2}\sqrt{1+\frac{4}{2-y}},\because D_{\mathrm{f}}=\left(-\infty,\frac{3}{2}\right)$

$\therefore \mathrm{f}^{-1}:x\mapsto\frac{3}{2}-\frac{1}{2}\sqrt{1+\frac{4}{2-x}},x\in\mathbb{R}$, $x<2$ or $x>6$

4

(i)

$$\mathbf{a} \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n}$$

$$\mathbf{a} \cdot \mathbf{n} - \mathbf{b} \cdot \mathbf{n} = 0$$

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{n} = 0$$

$$\overrightarrow{BA} \cdot \overrightarrow{ON} = 0$$

Hence, AB is perpendicular to ON.

(ii)

$|\mathbf{a} \times \mathbf{n}|$ represents

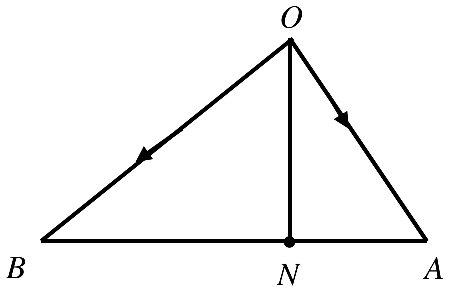
(1) twice area of triangle OAN (or)
(2) Area of parallelogram with adjacent sides OA and ON.

(iii)

$\left| \mathbf{a} \times \frac{\mathbf{n}}{|\mathbf{n}|} \right|$ represents

(1) the length of AN (or)

(2) (perpendicular) distance from A to ON



$$\tan 30° = \frac{AN}{|\mathbf{n}|}$$

$$AN = \frac{1}{\sqrt{3}} |\mathbf{n}|$$

(iv)

$$\tan 60° = \frac{NB}{|\mathbf{n}|}$$

$$NB = \sqrt{3} |\mathbf{n}|$$

$AN : NB$

$= \frac{1}{\sqrt{3}} |\mathbf{n}| : \sqrt{3} |\mathbf{n}|$

$= 1 : 3$

By Ratio Theorem,

$$\mathbf{n} = \frac{3\mathbf{a} + \mathbf{b}}{4}$$

5	(a) $x=\mathrm{e}^t-1, y=t-\mathrm{e}^t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=e^t$, $\frac{\mathrm{dy}}{\mathrm{d}t}=1-e^t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-e^t}{e^t}$ Eqn of normal: $y-\left(p-e^p\right)=-\ \frac{e^p}{1-e^p}(x-(e^p-1))$ When $x = 0$, $y-\left(p-e^p\right)=-\frac{e^p}{1-e^p}(0+1-e^p))$ $=-e^p$ $y=p-2e^p$ $Q(0,\ p-2e^p\ )$ $t-\mathrm{e}^t=-5$ From GC, $t=1.9368$ $\therefore\ \frac{\mathrm{dy}}{\mathrm{d}t}=1-e^{1.9368}$ $=\ -5.94\ \text{unit / s}$ (b) Let $x = \tan\ \theta$ $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$ When $x = 0$, $\tan\ \theta = 0 \Rightarrow \theta = 0$ When $x=\sqrt{3}, \tan\theta=\sqrt{3}\Rightarrow\theta=\frac{\pi}{3}$ $\text{Vol}=\pi\int_0^{\sqrt{3}}\frac{x^2}{(1+x^2)^2}\ \mathrm{d}x=\pi\int_0^{\frac{\pi}{3}}\frac{\tan^2\theta}{(1+\tan^2\theta)^2}\sec^2\theta\ \mathrm{d}\theta$ $=\pi\int_0^{\frac{\pi}{3}}\frac{\tan^2\theta}{(\sec^2\theta)^2}\sec^2\theta\ \mathrm{d}\theta$ $=\pi\int_0^{\frac{\pi}{3}}\frac{\tan^2\theta}{\sec^2\theta}\ \mathrm{d}\theta$ $=\pi\int_0^{\frac{\pi}{3}}\frac{\sin^2\theta}{\cos^2\theta}\div\frac{1}{\cos^2\theta}\ \mathrm{d}\theta$

	$=\pi\int_0^{\frac{\pi}{3}}\sin^2\theta\ \mathrm{d}\theta$ $=\pi\int_0^{\frac{\pi}{3}}\frac{1-\cos 2\theta}{2}\ \mathrm{d}\theta$ $=\frac{\pi}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{3}}$ $=\frac{\pi}{2}\left[\frac{\pi}{3}-\frac{1}{2}\sin 2\left(\frac{\pi}{3}\right)-0\right]$ $=\frac{\pi^2}{6}-\frac{\sqrt{3}}{8}\pi$
6	(i) $\mathrm{P}(X=0)=\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)(2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)+\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$ $=\frac{7}{20}$ (ii) $\mathrm{P}(X=2)$ $=\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)(2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ $=\frac{3}{20}$ $\mathrm{P}(X=1)$ $=1-\frac{3}{20}-\frac{7}{20}$ $=\frac{10}{20}$ Or $\mathrm{P}(X=1)$ $=\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)(2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(2)+\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)+\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)$ $=\frac{10}{20}$ Probability distribution of X <table><tr><td>x</td><td>0</td><td>1</td><td>2</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>$\frac{7}{20}$</td><td>$\frac{10}{20}$</td><td>$\frac{3}{20}$</td></tr></table> (iii) $\mathrm{E}(X)=0\times\frac{7}{20}+1\times\frac{10}{20}+2\times\frac{3}{20}=\frac{16}{20}=\frac{4}{5}$

	Var(X) $= 0^2 \times \frac{7}{20} + 1^2 \times \frac{10}{20} + 2^2 \times \frac{3}{20} - \left(\frac{16}{20}\right)^2 = \frac{23}{50}$ (iv) P(2 balls are of different colour|no heads) $= \dfrac{\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)(2)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\frac{7}{20}}$ $= \frac{3}{7}$ (v) Required probability = $\frac{7}{20} + \left[\left(\frac{3}{5}\right)\left(\frac{2}{4}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{4}\right)\right] - \left[\left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{2}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\right]$ $= \frac{11}{20}$
7	(i) No of ways $= \frac{{}^{10}C_5}{2} = 126$ (ii) No of ways $= 4! \times 5! \times 3! = 17280$ (iii) No of ways $= {}^{10}C_5 \times 5!$ $= 30240$ (iv) No of ways no two odd digit cards are next to each other $= (7-1)! \times {}^{7}P_3$ $= 151200$ Required probability = $\frac{151200}{362880} = \frac{5}{12}$
8	(i) Large number of glowing sticks with constant probability of 0.12 enables independence in selection. (ii) Let X be the number of green glowing sticks out of 8. $X \sim \mathrm{B}(8, 0.12)$ $\mathrm{P}(X \geq 3) = 1 - \mathrm{P}(X \leq 2)$ $= 0.060789...$ ≈ 0.0608 (3 s.f) (iii) Let Y be the number of packets that contain at least 3 glowing sticks out of 40 packets. $Y \sim \mathrm{B}(40, 0.060789)$ $\mathrm{P}(Y < 6) = \mathrm{P}(Y \leq 5)$ $= 0.96733...$ ≈ 0.967 (3 s.f)

(iv) Let W be the number of blue glowing sticks out of 8.
$W \sim \mathrm{B}(8\,,\,p)$

Since 3 is the modal number,

$\mathrm{P}(W=3) > \mathrm{P}(W=2)$

$\binom{8}{3} p^3(1-p)^5 > \binom{8}{4} p^2(1-p)^6$ --- (1)

$56p^3(1-p)^5 > 28p^2(1-p)^6$

$2p > 1-p$

$p > \frac{1}{3}$

At the same time,

$\mathrm{P}(W=3) > \mathrm{P}(W=4)$

$\binom{8}{3} p^3(1-p)^5 > \binom{8}{4} p^4(1-p)^4$ --- (2)

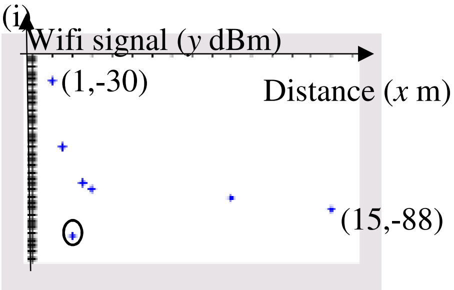
$56p^3(1-p)^5 > 70p^4(1-p)^4$

$\frac{4}{5}(1-p) > p$

$p < \frac{4}{9}$

Thus, $\frac{1}{3} < p < \frac{4}{9}$

9 (i)



There could be an obstruction which reduces the strength of the wifi signal or
There could be a wifi disruption.

(ii) The data points do not seem to lie on a straight line.

(iii) For $y = \frac{c}{x} + d$, $r = 0.97034... \approx 0.970$

For $y = \frac{e}{x^2} + f$, $r = 0.99733... \approx 0.997$

	Since the r value for the second model is closer to 1, $y=\frac{e}{x^2}+f$ is the better model. $e=55.81195...\approx 55.8\ (3\text{ s.f})$ $f=-85.42578...\approx -85.4\ (3\text{ s.f})$ (iv) When $x=5$, $y=\frac{55.81195}{5^2}-85.42578$ $=-83.193...\approx -83.2\ (3\text{ s.f})$ The required signal strength is -83.2 dBm This estimate is expected to be reliable as it is obtained by interpolation and $r\approx 0.997$ is close to 1.
10	(i) 'Keep the recorded values small since they are around 49' or 'give an indication of the variations around the hypothesized mean of 49. (ii) Unbiased estimate of μ, $\bar{x}=\frac{\sum(x-49)}{100}+49$ $=\frac{313}{100}+49$ $=52.13$ Unbiased estimate of σ^2, s^2 $=\frac{1}{100-1}\left(\sum(x-49)^2-\frac{\left[\sum(x-49)\right]^2}{100}\right)$ $=\frac{1}{99}\left(23280-\frac{(313)^2}{100}\right)$ $=225.26$ ≈ 225 (iii) $H_0:\mu=49$ $H_1:\mu\neq 49$ Under H_0, the test statistic is $Z=\frac{\overline{X}-\mu}{s/\sqrt{n}}\sim \mathrm{N}(0,1)$ approximately (by CLT) where μ = 49, $\bar{x}=52.13$, n = 100, $s^2=225.26$ p- value = 0.037027 ≈ 0.0370 Since the p-value < 0.05 (the significance level), we reject H_0 and conclude that at the 5% level, there is significant evidence that the Exam Board's claim is not valid.

	(iv) No assumptions needed since n = 100 is large, by Central Limit Theorem, the distribution of the sample mean marks is approximately normal. (v) $H_0 : \mu = 49$ $H_1 : \mu > 49$ Under H_0, the test statistic is $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0,1)$ where μ = 59.2, $n = 50$, $\sigma = 11.0$ Critical region: $z > 1.7507$ Since the mean mark of candidates has not increased, we do not reject $H_0 \Rightarrow$ $\dfrac{k-\mu}{\sigma / \sqrt{n}} < 1.7507$ $\dfrac{k-49}{11.0 / \sqrt{50}} < 1.7507$ $k < 51.723$ $\{k \in \mathrm{R} : k \le 51.7\}$
11	(i) Let X be the lifespan of a light bulb manufactured by OceanDoor $$X \sim \mathrm{N}\left(800, 70^2\right)$$ Let Y be the lifespan of a light bulb manufactured by DaRice $$Y \sim \mathrm{N}\left(700, 80^2\right)$$ $\overline{Y} = \dfrac{Y_1 + Y_2 + \cdots + Y_6}{6}$ $\overline{Y} \sim \mathrm{N}\left(700, \dfrac{80^2}{6}\right)$ $\mathrm{P}\left(\left\|\overline{Y} - \dfrac{1}{2}X\right\| \ge 200\right)$ $= 1 - \mathrm{P}\left(\left\|\overline{Y} - \dfrac{1}{2}X\right\| < 200\right)$ $= 1 - \mathrm{P}\left(-200 < \overline{Y} - \dfrac{1}{2}X < 200\right)$ $\overline{Y} - \dfrac{1}{2}X \sim \mathrm{N}\left(300, \dfrac{6875}{3}\right)$ $\mathrm{P}\left(\left\|\overline{Y} - \dfrac{1}{2}X\right\| \ge 200\right) = 0.982$ (3sf)

(ii)

$$\mathrm{P}\left(\overline{X} \le 825\right) \ge 0.99$$

$$\overline{X} \sim \mathrm{N}\left(800, \frac{70^2}{n}\right)$$

$$\mathrm{P}\left(Z \le \frac{825-800}{70/\sqrt{n}}\right) \ge 0.99$$

$$\mathrm{P}\left(Z \le \frac{5\sqrt{n}}{14}\right) \ge 0.99$$

$$\therefore \frac{5\sqrt{n}}{14} \ge 2.3263$$

$n \ge 42.4275$

Or:

When $n = 42$, $\mathrm{P}\left(\overline{X} \le 825\right) = 0.9897$

When $n = 43$, $\mathrm{P}\left(\overline{X} \le 825\right) = 0.9904$

Least $n = 43$

(iii)

$$X + Y \sim \mathrm{N}(1500, 11300)$$

$$\mathrm{P}(X + Y > 1450) = 0.681 \text{ (3 sf)}$$

(iv)

$$\mathrm{P}(X > 725 \text{ and } Y > 725)$$
$$= \mathrm{P}(X > 725) \times \mathrm{P}(Y > 725)$$
$$= 0.323753$$
$$= 0.324 \text{ (3 sf)}$$

(v)

Part (iii) include cases that part (iv) does not have.

(vi) The lifespan of all light bulbs are independent of each other.