Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/01

Paper 1 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)
Graph Paper

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	3	7	5	6	8	11	11	9	13	13	14	100

1 If x is a sufficiently small angle, find the first three non-zero terms in the Maclaurin series for $\sin^7(x+\frac{1}{4}\pi)$. [3]

2 (i) Using appropriate expansions from the List of Formulae (MF26), show that the series expansion of $\ln\left(1+e^{2x}\right)$ in ascending powers of x, up to and including the term in x^2, is $\ln 2 + x + \frac{x^2}{2}$. [3]

(ii) Find the set of values of x such that the value for the above expansion for $\ln\left(1+e^{2x}\right)$ is within ± 0.3 of the value of $\ln\left(1+e^{2x}\right)$. [2]

(iii) Deduce an expansion for $\dfrac{2}{1+e^{-2x}}$ up to and including the term in x. [2]

3 If $x = 3\cos^2\theta + 6\sin^2\theta$, show that $6 - x = 3\cos^2\theta$, and find a similar expression for $x-3$. By using the substitution $x = 3\cos^2\theta + 6\sin^2\theta$, evaluate exactly $\int_3^6 \dfrac{1}{\sqrt{(x-3)(6-x)}}\,dx$. [5]

4 Show that the following inequality

$$\tan x + \cot x > 4 \text{ for } 0 < x < \tfrac{1}{2}\pi$$

can be simplified to

$$0 < \sin 2x < \tfrac{1}{2}.$$ [4]

Hence solve exactly the inequality leaving your answer in terms of π. [2]

5 Functions f and g are defined by

$$\mathrm{f}: x \mapsto 1+\frac{2}{x-1}, \quad x \in \mathbb{R},\ x<1,$$

$$\mathrm{g}: x \mapsto \ln x, \quad x \in \mathbb{R},\ 0<x<1.$$

(i) Explain why the composite function gf does not exist. [1]

(ii) Find an expression for $\mathrm{fg}(x)$. Hence or otherwise, find $(\mathrm{fg})^{-1}(0)$. [4]

(iii) Find an expression for $\mathrm{h}(x)$ for each of the following cases:

(a) $\mathrm{gh}(x)=x$, [1]

(b) $\mathrm{hg}(x)=x^2+1$. [2]

6 **(a)** By using the substitution $y=zx^2$, find the general solution of the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x}=2xy-y^2, \quad \text{where } x \neq 0.$$

[4]

(i) Sketch the solution curve that passes through $(2,-4)$, indicating any stationary points and asymptotes clearly. [4]

(ii) State the particular solution for which y has no turning point. [1]

(b) A differential equation is of the form $\frac{\mathrm{d}y}{\mathrm{d}x}+y=px+q$, where p and q are constants. Its general solution is $y=4x-1+D\mathrm{e}^{-x}$, where D is an arbitrary constant. Find the values of p and q. [2]

7 **(a)** Given that $z^* = \dfrac{(2\mathrm{i})^3}{\left(\sqrt{3}+\mathrm{i}\right)^4}$, find the exact value of $|z|$ and $\arg(z)$.

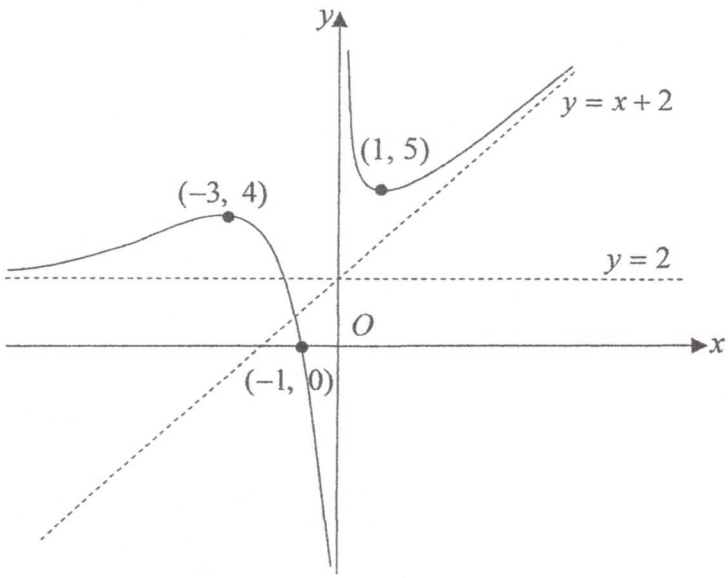
Hence state the smallest positive integer n such that z^n is purely imaginary. [5]

(b) When the polynomial $ax^4 + bx^3 + cx^2 + 24x - 44$, where a, b and $c \in \mathbb{R}$, is divided by $(x-1)$, $(x+1)$ and $(x-2)$, the remainders are -18, -54 and 0 respectively.

(i) Find the values of a, b and c. [3]

(ii) The equation $ax^4 + bx^3 + cx^2 + 24x - 44 = 0$, with the values of a, b and c found in part **(i)**, has a root $3 - (\sqrt{2})\mathrm{i}$. Find the other roots of the equation, showing your working clearly. [3]

8



The diagram shows the graph of $y = \mathrm{f}(x)$. The curve has turning points $(-3,\ 4)$ and $(1,\ 5)$ and crosses the x-axis at $(-1,\ 0)$. The curve has asymptotes $x = 0$, $y = 2$ and $y = x + 2$.

Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(|x|)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes.

Describe a sequence of three transformations which transforms the graph of $y = \mathrm{f}(x)$ to the graph of $y = \mathrm{f}(ax + b)$, where a and b are constants such that $a < -1$ and $0 < b < 1$. State the coordinates of the point where the graph of $y = \mathrm{f}(ax + b)$ cuts the x-axis. [4]

9 The line l_1 passes through the point A with the position vector $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and is parallel to $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}$, while the cartesian equation of the plane p is given by $tx-2y+z=-3$, where t is a real constant. It is known that l_1 and p have no point in common.

(i) Show that $t=-1$. [3]

(ii) Find the distance between l_1 and p. [2]

(iii) The line l_2 has the cartesian equation $2y=z,\ x=3$. Show that l_2 lies on p. [2]

(iv) Given that point B and point C lie on l_1 and l_2 respectively, find $\overrightarrow{BC}$ such that it is perpendicular to both l_1 and l_2. [3]

(v) Find the vector equation of the line of reflection of l_1 in p. [3]

10 Duncan, an aspiring marathon runner, embarks on the following training regime:

1. On Day 1, he runs 5 km.
2. On Day 2, he runs $(1+\alpha)$ times the distance ran in the previous day.
3. On Day 3, he runs α times the distance ran in the previous day.

You may take α to be a positive constant.

For each subsequent day, Duncan repeats parts 2 and 3 of the training regime, in that order. Thus on Day 4, he runs $(1+\alpha)$ times the distance ran on Day 3; and on Day 5, he runs α times the distance ran on Day 4, and so on.

(i) The distances that Duncan runs on odd-numbered days follows a geometric progression. State its common ratio in terms of α. [1]

(ii) By considering just the distances that Duncan runs on even-numbered days, find the range of values of α such that the total distance ran on all even-numbered days is finite. [2]

(iii) By considering the total distances ran on both odd and even numbered days, determine the theoretical maximum total distance, expressing your answer in terms of α. [2]

Duncan decides to fix $\alpha=0.65$.

(iv) (a) Show that the distance he ran on Day 10 is 10.915 km, correct to the nearest 0.001 km. [1]

(b) The distance he ran on Day n first exceeds 42.195 km. Find the value of n. [3]

(c) Duncan aims to complete his training regime on Day n, and instead of following the regime for that day, he plans to only run exactly 42.195 km on that day. Find the total distance that Duncan would have covered at the end of his training regime. [4]

11 In the study of Microeconomics, the price \$$p$ (in thousands) that consumers and producers of a particular product A is willing to pay to consume or produce x quantities (in thousands) of the product is modelled by the following equations:

- Producers (supply curve): $x = t + e^t - 1,\ p = t^2$, where $t > 0$.
- Consumers (demand curve): $\dfrac{xp}{10} + \sin^{-1}\left(\dfrac{p}{10}\right) = 1$, where $x > 0$ and $0 < p < 10$.

(i) The price p that consumers are willing to pay to consume x quantities of the product decreases as x increases. Use differentiation to verify this. [2]

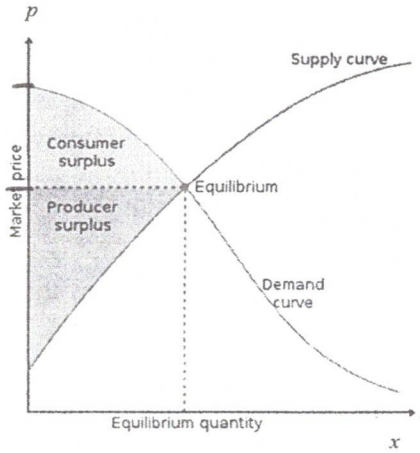
(ii) On the same diagram, sketch the demand and supply curves where x is the horizontal axis and p is the vertical axis. [3]

(iii) Market equilibrium is achieved when the quantity demanded and the quantity supplied are the same at a particular price. Find this price. [2]

Economic surplus, also known as total welfare or Marshallian surplus, refers to two related quantities:

- Consumer surplus is the monetary gain obtained by consumers because they are able to purchase a product for a price that is less than the highest price that they would be willing to pay. Thus the consumer surplus is the area of the region bounded by the demand curve, the p-axis and the horizontal line that passes through the equilibrium point.
- Producer surplus is the amount that producers benefit by selling at a market price that is higher than the least that they would be willing to sell for. Thus the producer surplus is the area of the region bounded by the supply curve, the p-axis and the horizontal line that passes through the equilibrium point.

The diagram below illustrates an example of the surpluses of **another product**.



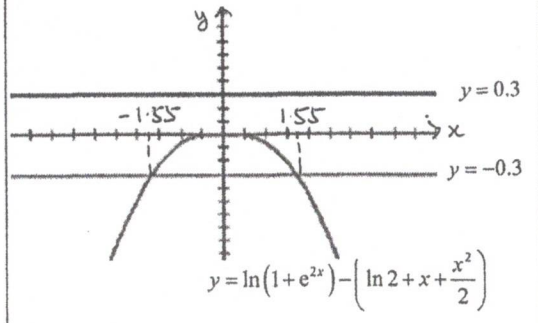
(iv) The total economic surplus is the sum of the consumer and producer surpluses at the market equilibrium. Find the total economic surplus for product A. [4]

(v) Due to a technological advancement in the manufacturing of product A, the supply curve is now translated a units in the positive x-direction and the new market equilibrium is achieved when $x = 5$. Find the value of a. [3]

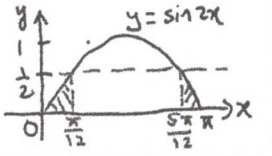
2018 Year 6 H2 Math Prelim Exam (Paper 1)
Solution & Markers' Comments

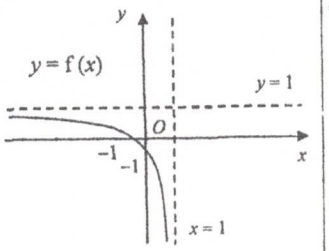
Suggested Solution	Markers' comments
$\sin^7(x+\frac{1}{4}\pi)$ $=(\sin x\cos\frac{1}{4}\pi+\cos x\sin\frac{1}{4}\pi)^7$ $=\left(\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}\right)^7$ $\approx\frac{1}{8\sqrt{2}}\left[x+\left(1-\frac{x^2}{2}\right)\right]^7$ $=\frac{1}{8\sqrt{2}}\left(1+\left(x-\frac{x^2}{2}\right)\right)^7$ $=\frac{1}{8\sqrt{2}}\left(1+7\left(x-\frac{x^2}{2}\right)+\frac{7\times6}{1\times2}\left(x-\frac{x^2}{2}\right)^2+\ldots\right)$ $\approx\frac{1}{8\sqrt{2}}\left(1+7x+\frac{35}{2}x^2\right)$	• In general, it's easier to use small angle approximation instead of repeated differentiation to find the Maclaurin series. • Common mistake is to assume that $(x+\frac{1}{4}\pi)$ is small. Even though x is small, do note that $\frac{1}{4}\pi$ **is not small.** Hence $\sin^7(x+\frac{1}{4}\pi)\neq(x+\frac{1}{4}\pi)^7$. See also Assign8 Q2.
	Total Marks: 3

Suggested Solution	Markers' comments
$\ln(1+e^{2x})=\ln\left(1+1+2x+\frac{(2x)^2}{2!}+\ldots\right)$ $=\ln(2+2x+2x^2+\ldots)$ $=\ln\left[(2)(1+x+x^2)\right]$ $=\ln 2+\ln\left[1+(x+x^2+\ldots)\right]$ $=\ln 2+(x+x^2+\ldots)-\frac{(x+x^2+\ldots)^2}{2}+\ldots$ $=\ln 2+x+\frac{x^2}{2}+\ldots$ (shown)	• The given instruction is to use appropriate expansions from MF26. Students who differentiated $\ln(1+e^{2x})$ to obtain the Maclaurin series is not given any credit. • Some students wrote $\ln(1+e^{2x})$ $=e^{2x}-\frac{(e^{2x})^2}{2}+\ldots$ $\approx\left[1+2x+\frac{(2x)^2}{2!}\right]-\frac{\left[1+2x+\frac{(2x)^2}{2!}\right]^2}{2}$ Firstly, the RHS will not produce an expression which involves ln 2 (Think of a **different approach** if your method does not lead to the correct answer). Secondly, we will not be able to correctly evaluate the constant term for $\ln(1+e^{2x})$ as the constant 1 will appear in every term of the expansion of e^{2x}.

(ii)	$-0.3<\ln(1+e^{2x})-\left(\ln 2+x+\frac{x^2}{2}\right)<0.3$ Using GC, $\{x\in\mathbb{R}:-1.55<x<1.55\}$  	• Quite a number of students did not leave their answer in set notation form as required by the question. • The use of GC to solve the inequality $-0.3<\ln(1+e^{2x})-\left(\ln 2+x+\frac{x^2}{2}\right)<0.3$ is expected since the question did not forbid the use of GC.
(iii)	$\because\ln(1+e^{2x})=\ln 2+x+\frac{x^2}{2}+\ldots$ Differentiate both sides wrt x: $\frac{2e^{2x}}{1+e^{2x}}=1+x+\ldots$ $\frac{2e^{2x}/e^{2x}}{(1+e^{2x})/e^{2x}}=1+x+\ldots$ ($\div$ by e^{2x} in num & denom) $\therefore\frac{2}{1+e^{-2x}}=1+x+\ldots$	• Many students either did not realise or were unable to use the result obtained in part (i) to **deduce** part (iii). The word deduce means that we need to use the result in part (i) to solve part (iii). • $[1+(1-2x)+\ldots]^{-1}$ $\neq 1+(-1)(1-2x)+\ldots$ • Likely forms of new expression o Derivative of the original o Integration of the original o Replacement of x by kx
		Total Marks: 7

Suggested Solution	Markers' comments
$x = 3\cos^2\theta + 6\sin^2\theta$ $6 - x = 6 - 3\cos^2\theta - 6\sin^2\theta$ $= 6(1-\sin^2\theta) - 3\cos^2\theta$ $= 6\cos^2\theta - 3\cos^2\theta$ $= 3\cos^2\theta$ (shown) $x - 3 = 3\cos^2\theta + 6\sin^2\theta - 3$ $= 3(\cos^2\theta - 1) + 6\sin^2\theta$ $= 3(-\sin^2\theta) + 6\sin^2\theta$ $= 3\sin^2\theta$	• Most students obtain the 2 marks. • Some students tried replacement, which is totally wrong since it is not a change of variable but simple trigonometry.
$x = 3\cos^2\theta + 6\sin^2\theta$, $\frac{dx}{d\theta} = [6\cos\theta(-\sin\theta) + 12\sin\theta(\cos\theta)]$ $= 6\sin\theta\cos\theta$ when $x = 3$, $3\cos^2\theta + 6\sin^2\theta = 3$ $3(\cos^2\theta - 1) + 6\sin^2\theta = 0$ $3(-\sin^2\theta) + 6\sin^2\theta = 0$ $\sin\theta = 0 \Rightarrow \theta = 0$ when $x = 6$, $3\cos^2\theta + 6\sin^2\theta = 6$ $3\cos^2\theta + 6(\sin^2\theta - 1) = 0$ $3\cos^2\theta + 6\cos^2\theta = 0$ $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $\int_3^6 \frac{1}{\sqrt{(x-3)(6-x)}}\,dx$ $= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{(3\sin^2\theta)(3\cos^2\theta)}}(6\sin\theta\cos\theta)\,d\theta$ $= \frac{6}{3}\int_0^{\frac{\pi}{2}} 1\,d\theta$ $= 2[\theta]_0^{\frac{\pi}{2}} = 2\left[\frac{\pi}{2} - 0\right]$ $= \pi$	• For definite integral involving substitution, remember to change $x = 3$ and 6 to the corresponding values in θ. • There were a lot of careless mistakes in manipulation. • Students should use GC to verify their final answer
	Total Marks: 5

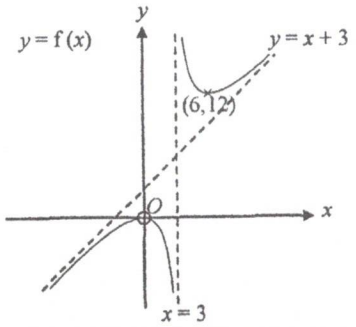
Qn	Suggested Solution	Markers' comments
4	$\tan x + \cot x > 4$ $\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} > 4$ $\frac{\sin^2 x + \cos^2 x}{\sin x\cos x} > 4$ $\frac{1}{\frac{1}{2}\sin 2x} > 4$ $\frac{1}{\sin 2x} > 2$ $\therefore 0 < \sin 2x < \frac{1}{2}$ (shown) So we have $0 < 2x < \frac{1}{6}\pi$ or $\frac{5}{6}\pi < 2x < \pi$ $0 < x < \frac{1}{12}\pi$ or $\frac{5}{12}\pi < x < \frac{1}{2}\pi$ 	• Always refer to the result to be shown for hints on how to start. Since the shown result contains only sin 2x, you should convert everything in terms of sin and cos (Note the pairing of trigo functions i.e. sin-cos, sec-tan, cosec-cot). • When dealing with inequalities that are not so straight-forward, it is always good to sketch graphs.
		Total Marks : 6

Qn	Suggested Solution	Markers' comments
5(i)	$R_f = (-\infty, 1) \not\subset D_g = (0, 1)$ Thus gf does not exist. 	• R_f and D_g need to be written explicitly as $R_f = (-\infty, 1)$ and $D_g = (0, 1)$ in order to convince that R_f is not a subset of D_g. • A handful of students did not get the correct R_f as they did not consider the restriction on the domain of f when it comes to the sketching of the graph of f.
(ii)	$fg(x) = f(\ln x) = 1 + \frac{2}{\ln x - 1} = \frac{\ln x + 1}{\ln x - 1}$	• This part is generally well done. Students need to know how to use **Method 1** to obtain the answer.

	Method 1 Let $x=(\mathrm{fg})^{-1}(0)$ $\mathrm{fg}(x)=0$ $\dfrac{\ln x+1}{\ln x-1}=0$ $\ln x+1=0$ $\ln x=-1$ $x=\mathrm{e}^{-1}$ $\therefore (\mathrm{fg})^{-1}(0)=\mathrm{e}^{-1}$ **Method 2 (work backwards from final value = 0)** Let $x=(\mathrm{fg})^{-1}(0)$ $\mathrm{fg}(x)=0$ Substitute $\mathrm{f}(x)=0\Rightarrow 0=1+\dfrac{2}{x-1}\Rightarrow x=-1$ Substitute $\mathrm{g}(x)=-1\Rightarrow -1=\ln x\Rightarrow x=\mathrm{e}^{-1}$ $\therefore (\mathrm{fg})^{-1}(0)=\mathrm{e}^{-1}$ $\{\mathrm{e}^{-1}\}\xrightarrow{\mathrm{g}}\{-1\}\xrightarrow{\mathrm{f}}\{0\}$ **Method 3 (find inverse directly)** Let $y=\mathrm{fg}(x)=\dfrac{\ln x+1}{\ln x-1}$ $y\ln x-y=\ln x+1$ $\ln x(y-1)=y+1$ $\ln x=\dfrac{y+1}{y-1}$ $x=\mathrm{e}^{\frac{y+1}{y-1}}$ $(\mathrm{fg})^{-1}(x)=\mathrm{e}^{\frac{x+1}{x-1}}$ $\therefore (\mathrm{fg})^{-1}(0)=\mathrm{e}^{\frac{0+1}{0-1}}=\mathrm{e}^{-1}$	
a)	$\mathrm{gh}(x)=x\Rightarrow \mathrm{h}(x)=\mathrm{g}^{-1}(x)=\mathrm{e}^{x}$	• Well-attempted
	Method 1 $\mathrm{hg}(x)=x^2+1$ $\mathrm{hg}\left(\mathrm{g}^{-1}(x)\right)=\left(\mathrm{g}^{-1}(x)\right)^2+1$ $\therefore \mathrm{h}(x)=\left(\mathrm{e}^{x}\right)^2+1=\mathrm{e}^{2x}+1$	• We cannot simply post-multiply both sides by g^{-1} to obtain $\mathrm{h}(x)$. However, we could use **Method 1** to replace x by $\mathrm{g}^{-1}(x)$ since $\mathrm{g}\left(\mathrm{g}^{-1}(x)\right)=x$ and $\mathrm{g}^{-1}(x)=\mathrm{e}^{x}$.

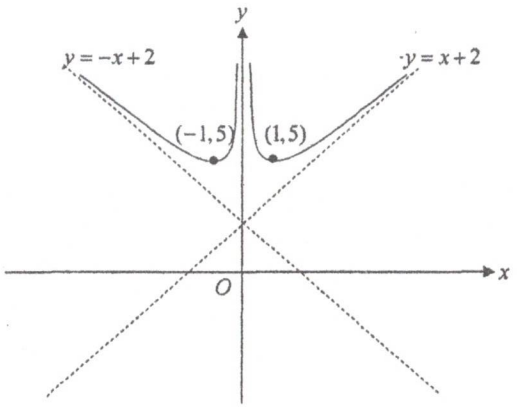
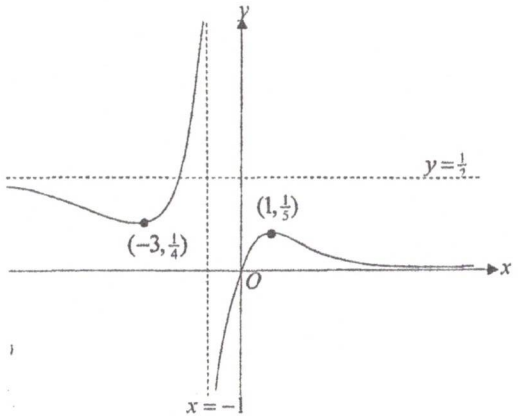
	Method 2 $\mathrm{hg}(x)=x^2+1$ $\mathrm{h}\left(\mathrm{g}(x)\right)=x^2+1$ $\mathrm{h}(\ln x)=\left(\mathrm{e}^{\ln x}\right)^2+1$ $\therefore \mathrm{h}(x)=\left(\mathrm{e}^{x}\right)^2+1=\mathrm{e}^{2x}+1$	
		Total Marks: 8

Qn	Suggested Solution	Markers' comments
6(a)	$x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}=2xy-y^2$ ...(1) Given $y=zx^2$: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2xz+x^2\dfrac{\mathrm{d}z}{\mathrm{d}x}$ ...(2) Substitute (2) into (1): $x^2\left(2xz+x^2\dfrac{\mathrm{d}z}{\mathrm{d}x}\right)=2x\left(zx^2\right)-\left(zx^2\right)^2$ $x^4\dfrac{\mathrm{d}z}{\mathrm{d}x}=-z^2x^4$ $\dfrac{\mathrm{d}z}{\mathrm{d}x}=-z^2$ $\int\dfrac{1}{z^2}\,\mathrm{d}z=-\int 1\,\mathrm{d}x$ $-\dfrac{1}{z}=-x+C$ $-\dfrac{x^2}{y}=-x+C$ $y=\dfrac{x^2}{x-C}$	• Many students failed to do chain rule to get equation (2), treating z as a constant. Students are reminded that if z is a constant with respect to x, then there is no change of variable involved. • There are some careless mistakes, ranging from sign error (which can be checked by simple differentiation and substituting back) to the following: $\dfrac{1}{z}=x-C$ $z=\dfrac{1}{x}-C$ (wrong, should be $z=\dfrac{1}{x-C}$)
(i)	Given $(2,-4)$, $-4=\dfrac{2^2}{2-C}\Rightarrow C=3$ Hence, $y=\dfrac{x^2}{x-3}=x+3+\dfrac{9}{x-3}$	• Many students who has the correct general equation only drew the bottom half of the graph. Before sketching, it would be good to have an idea of what the graph should look like so that you can spot the missing parts.

Suggested Solution	Markers' comments
	• Students can also use zoom-fit on their GC to see the whole graph. • Close to all students did not omit (0, 0), when it is given that $x \neq 0$. • There are some students who has the correct general equation but evaluated the oblique asymptote incorrectly.
When $C=0$, particular soln is $y=\frac{x^2}{x-0}=x$ which is a straight line and has no turning point.	• Many students didn't answer the question by stating the equation explicitly. • Many did differentiation instead or tried to solve $\frac{dy}{dx}\neq 0$. • Students are reminded to look at the number of mark(s) given for the time/effort needed.
Given $y=4x-1+De^{-x} \Rightarrow \frac{dy}{dx}=4-De^{-x}$ $\frac{dy}{dx}+y=\left(4-De^{-x}\right)+\left(4x-1+De^{-x}\right)=4x+3$ $\therefore p=4,\ q=3$	• Many students got this correct. • There were some students who attempted to integrate y in terms of x to obtain xy, which is completely incorrect.
	Total marks : 11

Qn	Suggested Solution	Markers' comments
7(a)	$z^*=\frac{(2i)^3}{\left(\sqrt{3}+i\right)^4}=\frac{-8i}{\left(\sqrt{3}+i\right)^4}$ $\lvert z\rvert=\lvert z^*\rvert$ $=\left\lvert\frac{-8i}{\left(\sqrt{3}+i\right)^4}\right\rvert=\frac{8}{\left(\sqrt{\left(\sqrt{3}\right)^2+1^2}\right)^4}$ $=\frac{8}{16}=\frac{1}{2}$ $\arg(z)=-\arg\left(z^*\right)$ $=-\arg\left(\frac{-8i}{\left(\sqrt{3}+i\right)^4}\right)$ $=-\left[\arg(-8i)-4\arg\left(\sqrt{3}+i\right)\right]$ $=-\left[-\frac{1}{2}\pi-4\left(\frac{1}{6}\pi\right)\right]$ $=\frac{7}{6}\pi$ $\therefore \arg(z)=\frac{7}{6}\pi-2\pi=-\frac{5}{6}\pi$ $\arg(z^n)=n\arg(z)=-\frac{5}{6}n\pi$ Since z^n is purely imaginary, $-\frac{5}{6}n\pi=(2k+1)\left(\frac{1}{2}\pi\right),\ k\in\mathbb{Z}$ $\Rightarrow n=-\frac{3}{5}(2k+1)$ $\therefore$smallest positive integer $n=3$ (when $k=-3$) **Alternative** $z^*=\frac{(2i)^3}{\left(\sqrt{3}+i\right)^4}=\frac{\left(2e^{i\frac{\pi}{2}}\right)^3}{\left(2e^{i\frac{\pi}{6}}\right)^4}$ $=\frac{8e^{i\frac{3\pi}{2}}}{16e^{i\frac{4\pi}{6}}}=\frac{1}{2}e^{i\left(\frac{3\pi}{2}-\frac{4\pi}{6}\right)}$ $=\frac{1}{2}e^{i\frac{5\pi}{6}}$ $\boxed{\therefore z=\frac{1}{2}e^{-i\frac{5\pi}{6}}}$ $\Rightarrow \lvert z\rvert=\frac{1}{2},\ \arg(z)-\frac{5}{6}\pi$ $z=re^{-i\theta}$ where $r=\lvert z\rvert=\frac{1}{2}$ $\theta=\arg(z)=-\frac{5}{6}\pi$	• Important to remember the properties of complex numbers: ✓ $\lvert z\rvert=\lvert z^*\rvert$ ✓ $\arg(z)=-\arg\left(z^*\right)$ • Not efficient to expand the denominator $\left(\sqrt{3}+i\right)^4$ • The alternative method is a useful and efficient way of finding the modulus and argument of complicated expressions at one go. • It's wrong to write $\frac{7}{6}\pi=-\frac{5}{6}\pi$. A separate line should be used for this. • On the other hand, for complex numbers which are purely real, arguments would take the form $k\pi, k\in\mathbb{Z}$
(b)(i)	Let $f(x)=ax^4+bx^3+cx^2+24x-44$ $f(1)=-18\Rightarrow a+b+c=2$ $f(-1)=-54\Rightarrow a-b+c=14$ $f(2)=0\Rightarrow 16a+8b+4c=-4$ From GC : $a=1, b=-6, c=7$	• Instead of long division, it's easier to use remainder and factor theorems to find the 3 equations, before solving simultaneously using GC.
(ii)	$x^4-6x^3+7x^2+24x-44=0$ If $3-(\sqrt{2})i$ is a root, $3+(\sqrt{2})i$ is also a root (since <u>equation has all real coefficients</u> OR by <u>conjugate root theorem</u>)	• It's important to state the <u>reason</u> for existence of the conjugate root $3+(\sqrt{2})i$. ✓ Eqn has all real coeff; or ✓ Conjugate root theorem

	Method 1 Observe $(x-2)$ is a **factor** of the polynomial equation. Compare product of last terms, $[x\boxed{-(3-(\sqrt{2})\mathrm{i})}][x\boxed{-(3+(\sqrt{2})\mathrm{i})}](x\boxed{-2})(x\boxed{+a}) = x^4-6x^3+7x^2+24x\boxed{-44}$ $(3-(\sqrt{2})\mathrm{i})(3+(\sqrt{2})\mathrm{i})(-2)(a) = -44$ $(3^2+(\sqrt{2})^2)(-2)a = -44$ $a = 2$ **Method 2** $[x-(3-(\sqrt{2})\mathrm{i})][x-(3+(\sqrt{2})\mathrm{i})] = [(x-3)+(\sqrt{2})\mathrm{i}][(x-3)-(\sqrt{2})\mathrm{i}]$ $= [(x-3)^2+2]$ $= x^2-6x+11$ Since $(x-2)$ is a **factor** of the polynomial equation, $x^4-6x^3+7x^2+24x-44=0$ $\Rightarrow (x^2-6x+11)(x-2)(x+2)=0$ (by inspection) $\therefore$ the other roots are $3+(\sqrt{2})\mathrm{i}$, 2 and -2	• Note the correct description ✓Factor : $(x-2)$ ✓Root : $x=2$ • Note that question states "**...showing your working clearly.**", so use of GC is not allowed to find the rest of the roots. But you can use the GC to verify your answers. • Conclude by listing the **other roots together.**
		Total marks : 11

Qn	Suggested Solution	Markers' comments
8(i)		• A number of students confused the graph of $y=\mathrm{f}(\lvert x\rvert)$ with the graph of $y=\lvert \mathrm{f}(x)\rvert$. • For the former, $y=\mathrm{f}(\lvert x\rvert)$ $=\begin{cases}\mathrm{f}(x) & \text{if } x\ge 0\\ \mathrm{f}(-x) & \text{if } x<0\end{cases}$ Thus the graph to be drawn is to retain the positive x-region and reflect it in the y-axis. • For the latter, $y=\lvert \mathrm{f}(x)\rvert$ $=\begin{cases}\mathrm{f}(x) & \text{if } \mathrm{f}(x)\ge 0\\ -\mathrm{f}(x) & \text{if } \mathrm{f}(x)<0\end{cases}$ Thus the graph to be drawn is to retain the positive y-region and reflect the negative y-region in the x-axis.
(ii)		• Most were able to label the critical features of the graph (Stationary Points, Asymptotes) though quite a number fail to observe that the graph passes through the origin. • For a step to step guide on how to sketch the reciprocal graph, scan the QR code below: (Can also be accessed via team drive)

1. Translate the graph by b units in the negative x-direction. 2. Scale the graph by a factor of $-\frac{1}{a}$ parallel to the x-axis. (Also accept scale factor of $\frac{1}{\lvert a\rvert}$. But do not accept scale factor of $\frac{1}{a}$.) 3. Reflect in the y-axis. From original graph, $f(-1)=0$. To find x-intercept of new graph, $f(ax+b)=0$ so $ax+b=0$ $\therefore x=\frac{-1-b}{a}$ Coordinates of the point where the graph of $y=f(ax+b)$ cuts the x-axis is $(\frac{-1-b}{a},0)$ or $(\frac{1+b}{\lvert a\rvert},0)$	• Note that for each transformation, only x or y is to be replaced. • Recommended order of transformation of graphs of the form $y=f(x)$ is TSR for x, RST for y while for graphs of the form $g(y)=f(x)$ e.g. $x^2+y^2=1$ is TSR for both x and y. • Since $a<0$, you need to be careful when performing the scaling. The scale factor should be the positive constant $-\frac{1}{a}$. • Note that for this equation, transformations only involved x, the x-intercept in new graph must result from the transformation of the original x-intercept.
	Total Marks: 9

	Suggested Solutions	Markers' comments
	l_1: $\underset{\sim}{r}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}$, $\lambda\in\mathbb{R}$ $p: tx-2y+z=-3 \Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}=-3$ The direction vector of l_1 is parallel to p (i.e., perpendicular to the normal of p) and the point A does not lie on p. $\begin{pmatrix}t\\t^2+1\\3\end{pmatrix}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}=0 \Rightarrow t^2-2t^2-2+3=0 \Rightarrow t=-1$ or 1	• If the line and the plane have no common point of intersection, the line has to be parallel to the plane and not lying on the plane. Hence have to show that the line is perpendicular to the normal of the plane, and that any point on the line does not lie on the plane. • Since it is a "show" question, sufficient working is expected.

	$\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}t\\-2\\1\end{pmatrix}\neq-3 \Rightarrow t-6+2\neq-3 \Rightarrow t\neq1$ Therefore, $t=-1$ (shown)	
(ii)	The distance between l_1 and p = shortest distance of A to p $=\frac{\lvert d-\overrightarrow{OA}\cdot\underset{\sim}{n}\rvert}{\lvert\underset{\sim}{n}\rvert}$ $=\frac{\left\lvert -3-\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}\right\rvert}{\sqrt{6}}$ $=\frac{2}{\sqrt{6}}$	• This is finding the length of projection of the vector (joining point A and any point on the plane) onto the normal of the plane. • Instead of using $\overrightarrow{OA}\cdot\underset{\sim}{n}$, some students use $\underset{\sim}{d}_{l_1}\cdot\underset{\sim}{n}$ which is incorrect.
(iii)	l_2: $2y=z, x=3, \Rightarrow \underset{\sim}{r}=\begin{pmatrix}3\\0\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\2\end{pmatrix}, \mu\in\mathbb{R}$ **Method 1** $\begin{pmatrix}0\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}=0-2+2=0$ Therefore, l_2 is parallel to p. $\begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$ Therefore, l_2 lies on p (shown). **Method 2** Substitute l_2: $2y=z, x=3$ into equation of p: LHS of $p=-3-2y+2y=-3=$ RHS of p	• Most students get this part correct. • Take note of proper presentation.

Method 1

$\overrightarrow{OB}=\begin{pmatrix}1-\lambda\\3+2\lambda\\2+3\lambda\end{pmatrix}$, for some $\lambda\in\mathbb{R}$

$\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}$, for some $\mu\in\mathbb{R}$

$\overrightarrow{BC}=\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}$

Since BC is perpendicular to both l_1 and l_2,

$\overrightarrow{BC}\bullet\underset{\sim}{d}_{l_1}=0$ and $\overrightarrow{BC}\bullet\underset{\sim}{d}_{l_2}=0$

$\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\bullet\begin{pmatrix}-1\\2\\3\end{pmatrix}=0$ and $\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\1\\2\end{pmatrix}=0$

$7\lambda-4\mu=7\quad\cdots(1)$ $\qquad 8\lambda-5\mu=-7\quad\cdots(2)$

Solve (1) and (2): $\mu=-\frac{7}{3}$ and $\lambda=-\frac{7}{3}$

$\overrightarrow{OB}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\left(-\frac{7}{3}\right)\begin{pmatrix}-1\\2\\3\end{pmatrix}=\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\2\mu\end{pmatrix}=\begin{pmatrix}3\\-\frac{7}{3}\\-\frac{14}{3}\end{pmatrix}$

$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}3\\-\frac{7}{3}\\-\frac{14}{3}\end{pmatrix}-\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$

Method 2

$\underset{\sim}{d}_{l_1}\times\underset{\sim}{d}_{l_2}=\begin{pmatrix}1\\-2\\-3\end{pmatrix}\times\begin{pmatrix}0\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

$\overrightarrow{BC}=\begin{pmatrix}3-1+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}=\begin{pmatrix}2+\lambda\\\mu-3-2\lambda\\2\mu-2-3\lambda\end{pmatrix}=\beta\begin{pmatrix}-1\\-2\\1\end{pmatrix}$

Solving the equation, $\beta=\frac{1}{3},\mu=\lambda=-\frac{7}{3}$

Hence $\overrightarrow{BC}=\frac{1}{3}\begin{pmatrix}-1\\-2\\1\end{pmatrix}$.

- Since point B and C lie on l_1 and l_2 respectively, we can formulate the vector expression for $\overrightarrow{BC}$.
- Also, since $\overrightarrow{BC}$ is perpendicular to both lines, it would be perpendicular to the director vector of both lines.
- Alternatively, $\overrightarrow{BC}$ is parallel to the normal vector of both lines.
- Be careful when solving for the values of μ and λ using your GC.

(v)

Method 1

Let l_1' be the line of reflection of l_1 in p.

Let point B' be the point lying on l_1' and is also the reflection of B in p.

$\underset{\sim}{d}_{l_1'}=\underset{\sim}{d}_{l_1}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$ because l_1 is parallel to p.

$\overrightarrow{BB'}=2\overrightarrow{BC}\Rightarrow\overrightarrow{OB'}=2\overrightarrow{BC}+\overrightarrow{OB}$

$=2\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}+\begin{pmatrix}\frac{10}{3}\\-\frac{5}{3}\\-5\end{pmatrix}$

$=\begin{pmatrix}\frac{8}{3}\\-3\\-\frac{13}{3}\end{pmatrix}$

Equation of line of reflection $l_1':\underset{\sim}{r}=\begin{pmatrix}\frac{8}{3}\\-3\\-\frac{13}{3}\end{pmatrix}+\gamma\begin{pmatrix}-1\\2\\3\end{pmatrix},\ \gamma\in\mathbb{R}$

Method 2

Let point F be the foot of perpendicular of point A on plane p.

$\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

Since point F lies on plane p,

$\begin{pmatrix}1-\lambda\\3-2\lambda\\2+\lambda\end{pmatrix}\bullet\begin{pmatrix}-1\\-2\\1\end{pmatrix}=-3$

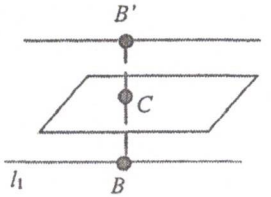
Solving, $\lambda=\frac{1}{3}$

$\overrightarrow{OF}=\begin{pmatrix}1\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\\frac{7}{3}\\\frac{7}{3}\end{pmatrix}$

$\overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}\Rightarrow\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$

$\overrightarrow{OA'}=2\begin{pmatrix}\frac{2}{3}\\\frac{7}{3}\\\frac{7}{3}\end{pmatrix}-\begin{pmatrix}1\\3\\2\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\\\frac{5}{3}\\\frac{8}{3}\end{pmatrix}$

- Recognize that l_1 is parallel to the plane, and so the reflection of the line will also be parallel to the plane.
- Then it suffices to find the point of reflection of any point on l_1 in the plane.
- Since point B lies on l_1, and point C lies on the plane (because point C lies on l_2 which lies on the plane), and that BC is perpendicular to the plane, $\overrightarrow{BB'}=2\overrightarrow{BC}$.



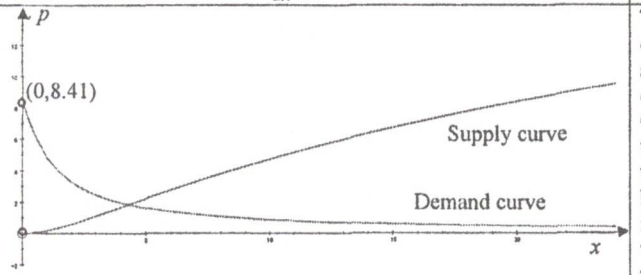
- It is also possible to find the point of reflection of point A using ratio (or mid-point) theorem. See **Method 2**.

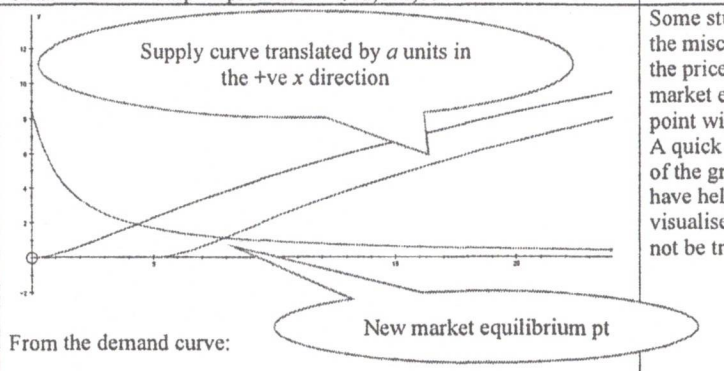
	Equation of line of reflection $l_1' : \underset{\sim}{r} = \begin{pmatrix} \frac{1}{3} \\ \frac{5}{3} \\ \frac{8}{3} \end{pmatrix} + \gamma \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \ \gamma \in \mathbb{R}$	
		Total marks: 13

	Solution	**Markers' comments**
	<table><tr><th></th><th>For first Odd-no Days</th><th>For first Even-no Days</th></tr><tr><td>1</td><td>5</td><td>$5(1+\alpha)$</td></tr><tr><td>2</td><td>$5(1+\alpha)(\alpha)$</td><td>$5(1+\alpha)^2(\alpha)$</td></tr><tr><td>3</td><td>$5(1+\alpha)^2(\alpha)^2$</td><td>$5(1+\alpha)^3(\alpha)^2$</td></tr><tr><td>4</td><td>$5(1+\alpha)^3(\alpha)^3$</td><td>$5(1+\alpha)^4(\alpha)^3$</td></tr><tr><td>5</td><td>$5(1+\alpha)^4(\alpha)^4$</td><td>$5(1+\alpha)^5(\alpha)^4$</td></tr><tr><td></td><td>$\vdots$</td><td>$\vdots$</td></tr></table> The common ratio for both sequences $= \alpha(1+\alpha)$	Majority are able to get this mark.
	For GP sum to infinity to exist, need $-1 < \alpha(1+\alpha) < 1$ Since distance for each day is positive, we solve $0 < \alpha(1+\alpha) < 1$ From GC, $0 < \alpha < 0.618$ (3 sf) Since the question did not ask for exact answers, we can just use GC to solve!	Common Errors: 1. Wrong condition for GP sum to infinity to exist. 2. Forgot the condition $\alpha > 0$. 3. Errors in concluding the correct region in solving inequalities.
	Theoretical max total distance $= \dfrac{5}{1-\alpha(1+\alpha)} + \dfrac{5(1+\alpha)}{1-\alpha(1+\alpha)}$ $= \dfrac{5+5(1+\alpha)}{1-\alpha(1+\alpha)}$ $= \dfrac{10+5\alpha}{1-\alpha-\alpha^2}$ Sum to infinity for the total distance run on both even and odd days. Calculation errors are very common.	Refer to 2011 GCE A Level P1 Q9(ii) for similar phrasing. Students who applied the sum to infinity formula are more successful as compared to those who derived the sum to infinity from S_n.
	Distance on Day 10 $= 5(1.65)\left[(1.65)(0.65)\right]^4 = 10.915$ km	Students did well for this part.

(b)	**Method 1** For the distance on Day n to **first** exceed 42.195 km, observe that n must be even, where the distance is larger than the previous day. $5(1.65)\left[(1.65)(0.65)\right]^{\frac{n-2}{2}} > 42.195$, where n is **even** You should argue for the smallest **even** n in your presentation. From GC, <table><tr><td>n</td><td>Distance on Day n</td></tr><tr><td>48</td><td>$41.26595 < 42.195$</td></tr><tr><td>50</td><td>$44.25773 > 42.195$</td></tr></table> Smallest $n = 50$ **Method 2** Distance run on the mth odd-numbered day $= 5\left[(1.65)(0.65)\right]^{m-1}$ Distance run on the mth even-numbered day $= 5(1.65)\left[(1.65)(0.65)\right]^{m-1}$ Consider $5\left[(1.65)(0.65)\right]^{m-1} > 42.195$ vs $5(1.65)\left[(1.65)(0.65)\right]^{m-1} > 42.195$ $m \geq 32$ vs $m \geq 25$ Since m for the even-numbered days is smaller, the distance will first exceed 42.195 on Day $n = 25 \times 2$, i.e. smallest $n = 50$.	Common Errors: 1. Misinterpreted the question and constructed inequality involving the total distance. 2. Students forgot that this distance formula is only applicable for even n and concluded smallest $n = 49$. 3. Students forgot to consider the difference between finding the mth even-numbered day versus finding Day n. A lot concluded $n = 25$.
(c)	Distance for first 50 days = Distance for first 25 odd-numbered days + Distance for first 24 even-numbered days + Distance on Day 50 <table><tr><th></th><th>For first Odd-no Days</th><th>For first Even-no Days</th></tr><tr><td>1</td><td>5</td><td>$5(1+\alpha)$</td></tr><tr><td>2</td><td>$5(1+\alpha)(\alpha)$</td><td>$5(1+\alpha)^2(\alpha)$</td></tr><tr><td>3</td><td>$5(1+\alpha)^2(\alpha)^2$</td><td>$5(1+\alpha)^3(\alpha)^2$</td></tr><tr><td>4</td><td>$5(1+\alpha)^3(\alpha)^3$</td><td>$5(1+\alpha)^4(\alpha)^3$</td></tr><tr><td>5</td><td>$5(1+\alpha)^4(\alpha)^4$</td><td>$5(1+\alpha)^5(\alpha)^4$</td></tr><tr><td></td><td>$\vdots$</td><td>$\vdots$</td></tr><tr><td>Total distance</td><td>first 25 odd-numbered days $\dfrac{5\left\{\left[(1.65)(0.65)\right]^{25}-1\right\}}{(1.65)(0.65)-1}$</td><td>first 24 even-numbered days $\dfrac{5(1.65)\left\{\left[(1.65)(0.65)\right]^{24}-1\right\}}{(1.65)(0.65)-1}$</td></tr></table>	Common errors: 1. Counting the even- and odd-numbered days wrongly. 2. Didn't take into account the distance run on the last day 3. Error in applying S_n formula. Note that in $S_n = \dfrac{a(1-r^n)}{1-r}$, n represents the number of terms that you sum.

	Total distance covered $=\frac{5}{(1.65)(0.65)-1}\left\{\left[(1.65)(0.65)\right]^{25}-1\right\}$ $+\frac{5(1.65)}{(1.65)(0.65)-1}\left\{\left[(1.65)(0.65)\right]^{24}-1\right\}+42.195$ $=866.681$ km	
		Total marks: 13

	Suggested Solution	**Markers' comments**
	$\frac{xp}{10}+\sin^{-1}\left(\frac{p}{10}\right)=1$ Differentiate wrt to x, $\frac{p}{10}+\frac{x}{10}\frac{dp}{dx}+\frac{1}{10\sqrt{1-\left(\frac{p}{10}\right)^2}}\left(\frac{dp}{dx}\right)=0$ Note: Remember to include this term $\frac{dp}{dx}=-\frac{p}{x+\frac{1}{\sqrt{1-\left(\frac{p}{10}\right)^2}}}$ You need to explain why $\frac{dp}{dx}<0$. Since $x>0$ and $0<p<10$, $\frac{dp}{dx}<0$. (shown)	
		To draw the demand curve, make x the subject in terms of p, draw this as Y1 in the GC and you can use the "Draw" feature in the GC (the prgm button) to draw the inverse curve of Y1 which will give you the p vs x curve. Note: You need to draw the shape of the supply curve with a point of inflexion.
	$\frac{(t+e^t-1)t^2}{10}+\sin^{-1}\left(\frac{t^2}{10}\right)=1$	Take note that if the qn did not mention anything about not

	Using GC, $t=1.3713$, $p=1.3713^2=1.8805=1.88$ (to 3 s.f) At market equilibrium, the price is \$1880.	being able to use the GC means that you can use GC to find the soln.
(iv)	Total economic surplus for product A $=\int_{1.8805}^{8.4147}\frac{10}{p}\left(1-\sin^{-1}\left(\frac{p}{10}\right)\right)dp+\int_0^{1.3713}2t(t+e^t-1)\,dt$ $=12.8$ Alternatively, $=\int_{1.8805}^{8.4147}\frac{10}{p}\left(1-\sin^{-1}\left(\frac{p}{10}\right)\right)dp+\int_0^{1.8805}(\sqrt{p}+e^{\sqrt{p}}-1)\,dp$ $=12.8$ Total economic surplus product A = \$12,800,000	Again, qn did not exclude the use of the GC.
(v)	 From the demand curve: when $x=5$, from the demand curve, $p=1.6654$ From the translated supply curve: $p=t^2=1.6654 \Rightarrow t=\sqrt{1.6654}=1.2905$ $x=t+e^t-1+a=5 \Rightarrow a=1.07$	Some students have the misconception that the price of the new market equilibrium point will not change. A quick mental sketch of the graph would have helped them to visualise that it will not be true.
		Total marks: 14

Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9758/02

Paper 2 September 2018

3 hours

Additional Materials: Answer Paper
Graph Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	5	9	6	8	12	8	9	9	10	12	12	100

Section A: Pure Mathematics [40 marks]

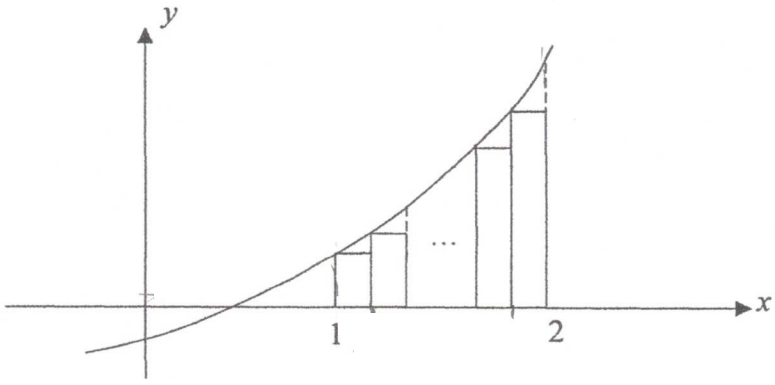
1 With reference to the origin O, the position vectors of point A and B are $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors.

(i) State the shortest distance from B to the line passing through O and A. [1]

(ii) Given that $\mathbf{c} = \mathbf{a} + \mathbf{b}$, state the geometrical meaning of $|\mathbf{b} \times \mathbf{c}|$ and show that $|\mathbf{b} \times \mathbf{c}| = |\mathbf{a} \times \mathbf{b}|$. [2]

(iii) Given that $\mathbf{a} \times 2\mathbf{b} = \mathbf{d} \times 3\mathbf{a}$, find a linear relationship between $\mathbf{a}, \mathbf{b}$ and $\mathbf{d}$. [2]

2 The equation of curve C_1 is given by $y = 3^x - 2$. The graph of C_1 is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$, where n is an integer, are drawn under C_1 for $1 \le x \le 2$.



(i) Show that the total area of all n rectangles, S_n, is given by

$$S_n = -2 + \frac{3}{n}\sum_{r=1}^{n} 3^{\frac{r-1}{n}}.$$

Hence evaluate S_n, leaving your answer in terms of n. [4]

(ii) Find the exact value of $\lim_{n \to \infty} S_n$. [2]

The equation of curve C_2 is given by $y = \dfrac{x^3}{\sqrt{4 + x^2}}$.

(iii) The region R is bounded by $C_1, C_2, x = 1$ and $x = 2$. Find the volume of the solid of revolution formed when R is rotated through 4 right angles about the x-axis, giving your answer correct to 2 decimal places. [3]

3 A sequence u_n is given by

$$u_n = \frac{3}{M-n+1}, \text{ where } n \text{ and } M \text{ are positive integers such that } n < M+1.$$

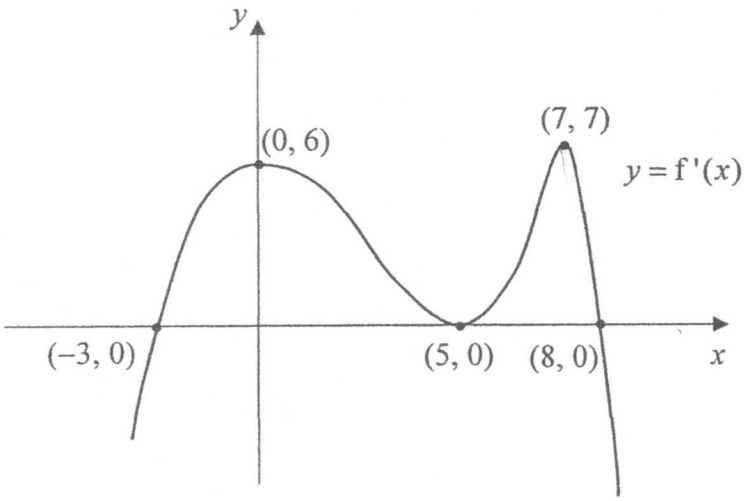
(i) Describe the behaviour of the sequence. [1]

Let S_n denote the sum of the first n terms of u_n.

(ii) Write down S_1, S_2 and S_3. Hence find $\sum_{n=1}^{M} S_n$ in terms of M. [3]

(iii) Show that $S_n > \frac{3n}{M}$. [2]

4 The diagram below shows the curve $y = \mathrm{f}'(x)$. It has turning points at (0, 6), (5, 0) and (7, 7) and intersects the x-axis at (−3, 0) and (8, 0).



(i) State the x-coordinates of the stationary points for the curve $y = \mathrm{f}(x)$. Hence determine the nature of these stationary points. [3]

(ii) State the range of values of x such that the curve $y = \mathrm{f}(x)$ is concave downwards. [2]

(iii) A student makes a claim on each of the following statements:

(A) If the curve $y = \mathrm{f}(x)$ passes through the point (−3, 7), then $\mathrm{f}(x) > 7$ for $-3 < x < 5$.

(B) By using integration by parts, $\int_{-3}^{8} x\mathrm{f}(x)\,\mathrm{d}x > 32\mathrm{f}(8) - \frac{9}{2}\mathrm{f}(-3)$.

For each of the above statements, explain briefly whether the student is right to make such a claim. [3]

5 (a) A curve has the equation $(x+y)^2 = 4e^{xy}$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(ii) Given that the curve cuts the positive y-axis at point A, find the equation of the tangent to the curve at A. [2]

(iii) The tangent to the curve at A meets the curve at another point B. Find the coordinates of B. [3]

(b) A closed cylinder is designed to contain a fixed volume of p cm^3 of liquid such that its external surface area is a minimum. Find the radius of the cylinder in terms of p in cm. [5]

Section B: Probability and Statistics [60 marks]

6 A test consists of 15 multiple choice questions, where each question has n possible options, of which only one is correct. A student took the test by randomly choosing an answer to each question. It is known that the probability of answering exactly 3 questions correctly is the same as the probability of answering exactly 4 questions correctly.

(i) By forming an equation in terms of n, find the value of n. [3]

Each correct answer is awarded 3 marks and each incorrect answer carries a penalty of 1 mark. The score is the total marks awarded based on the number of correct and incorrect answers.

(ii) Find the expected score, s, obtained by the student. [3]

(iii) Find the probability that the score obtained by the student is within 4 marks of s. [2]

7 During a symposium, 4 boys and 8 girls are divided into 4 groups of three each for discussion. How many ways are there to divide the 12 participants such that each group consists of exactly 1 boy? [2]

After the discussion, all members of the 4 groups sit at random at a round table.

Find the probability that

(i) the 3 members in each group are next to each other, [2]

(ii) every boy is separated from each other by exactly 2 girls. [3]

For the 12 participants, events A and B are defined by

A: every boy is separated from each other by exactly 2 girls; and
B: none of the boys are seated next to each other.

(iii) Determine if the events A and B are independent. [2]

8 An interactive simulation ride allows a group of 5 riders to take the ride at a time. The ride time, X minutes, follows a normal distribution with mean μ minutes and standard deviation 2 minutes. The ride starts promptly at 10 am daily with no wait time between any groups of 5 riders. There are only 4 scheduled rides every morning. At 10 am on a particular morning, there are already 20 people queuing for the ride. It is assumed that all the people in the queue will take the ride based on the sequence of the queue and the ride times are independent.

(i) Show that $\mu = 14$, correct to the nearest integer, if $\mathrm{P}(\mu < X < 16) = 0.35$. [2]

For the rest of the question, use $\mu = 14$ for your calculations. A ride is considered long if it has a ride time of at least 15 minutes.

(ii) Find the probability that the 12th person in the queue took the ride before 10.30 am on that morning. [3]

(iii) Show that the probability of having at least 2 long rides on that morning is 0.363. [2]

(iv) Given that there are at least 2 long rides on that morning, find the probability that none of these long rides are consecutive. [2]

9 To study the recent relationship between the property price index, p (in %) and the stock index, s (in thousands), of a particular city, Hilton recorded the readings from each of the past 8 quarters in the table below.

Stock Index, s (thousands)	2.12	2.53	2.63	2.70	2.75	2.83	2.87	2.98
Property Price Index, p (%)	115	130	145	140	146	150	155	170

Hilton realised that he recorded one of the values of p incorrectly.

(i) Sketch a scatter diagram for the data and circle the erroneous point X on your diagram. [2]

[For the remaining parts of this question, you should exclude the point X.]

Hilton proposes that s and p can be modelled by one of the formulae:

$$p = a + b s^5 \quad \text{or} \quad \ln p = c + d s,$$

where a, b, c and d are positive constants.

(ii) Determine the better model for the given data, giving a reason for your choice. [3]

(iii) Assuming that the value of s at X is correct, estimate the corresponding value of p. Give two reasons why you would expect this estimate to be reliable. [4]

Hilton concludes that higher stock index will lead to higher property price index. Comment on his conclusion in the context of the question. [1]

10 The discrete random variable X has probability mass function given by

$$\mathrm{P}(X=x)=\begin{cases}\dfrac{ax}{n(n+1)} & \text{for } x=1,2,\ldots,n,\\ 0 & \text{otherwise,}\end{cases}$$

where a is a constant.

(i) Show that $a=2$. [2]

(ii) Find $\mathrm{E}(X)$ in terms of n. You may use the result that $\sum_{r=1}^{n} r^2=\frac{n}{6}(n+1)(2n+1)$. [3]

Two players P_1 and P_2 play a game with $n+1$ tokens labelled 1 to $n+1$. Each player randomly picks one token without replacement and the player who picks the token with the smaller number loses. The amount of money lost by the losing player, in dollars, will be the number on the winning token. For example, if P_1 and P_2 pick the tokens labelled 5 and 3 respectively, P_2 loses \$5 to P_1.

(iii) Explain why the probability of P_2 losing a game is 0.5. [1]

(iv) Given that P_2 loses, find the probability P_2 loses \$$(m+1)$ in terms of m and n, where m is such that $1 \le m \le n$. [3]

(v) Using the result in part **(i)**, when P_2 loses a game, find the amount that he is expected to lose in terms of n. [3]

11 The speeds of cars along a busy stretch of road follow a normal distribution. Studies show that a mean speed of 50 km/h is needed to ensure a smooth flow of traffic. When the mean speed falls below 50 km/h, it may lead to road congestion. If this happens, Wireless Road Pricing (WRP) will be used to charge motorists to discourage them from using the road, hence improving the traffic condition. A random sample of the speeds, x km/h, of 120 cars along the stretch of road is recorded and the data are summarised by

$$\sum x=5415,\quad \sum x^2=351500.$$

(i) Calculate unbiased estimates of the population mean and variance of the speeds of the cars. [2]

(ii) What do you understand by the term 'unbiased estimate'? [1]

(iii) Test at the 3% significance level whether WRP is needed. [4]

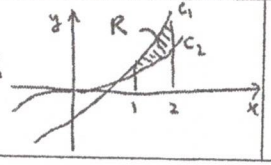
The Road Transport Authority decides to implement WRP on the same stretch of road. After the implementation, the speeds of a second sample of 80 cars are recorded with a mean speed of 60 km/h and a variance of 1100 (km/h)^2.

(iv) A test at the 8% significance level of the second sample suggests that the mean speed has increased beyond μ_0. Use an algebraic method to find the maximum value of μ_0. [4]

(v) State one assumption used in obtaining the sample statistics for the second sample. [1]

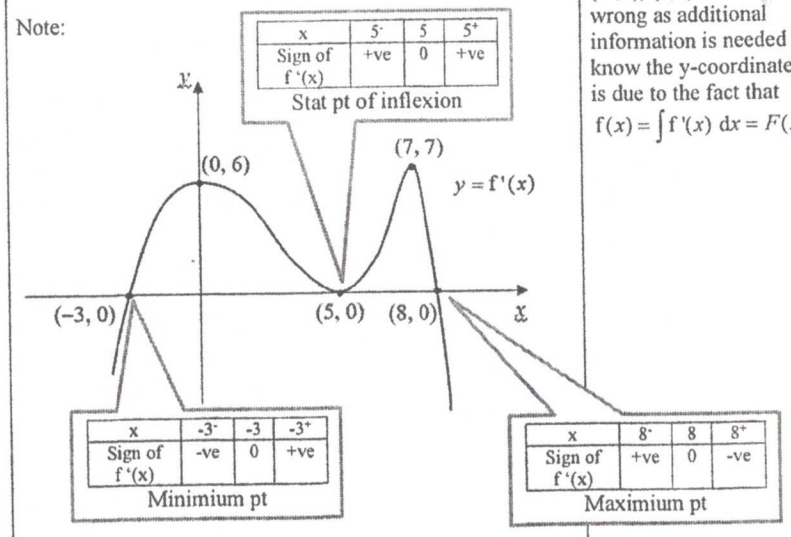
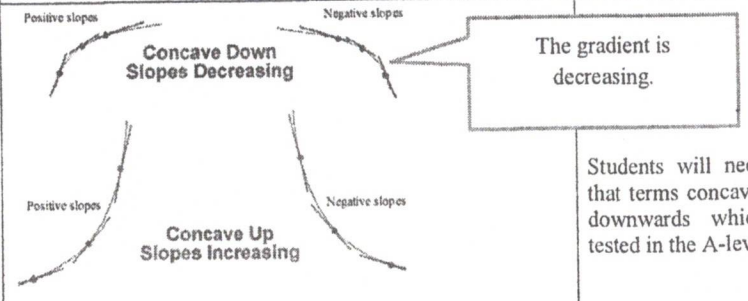
2018 Year 6 H2 Math Prelim (Paper 2)
Solution & Markers' Comments

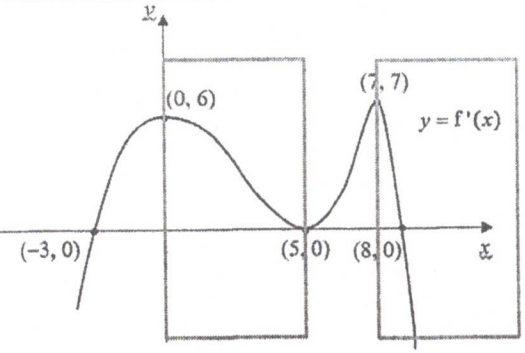
	Suggested Solution	Markers' comments
	Shortest distance $= \lvert \mathbf{b}\times\hat{\mathbf{a}}\rvert$ or $\dfrac{\lvert \mathbf{b}\times\mathbf{a}\rvert}{\lvert \mathbf{a}\rvert}$	• The key word here is 'state' => answer should be quite straightforward. • Quite a handful of students omitted the modulus sign, considered dot product instead or did not consider the unit vector for **a**.
	$\lvert \mathbf{b}\times\mathbf{c}\rvert$ is the area of a parallelogram with sides OB and OC. $\lvert \mathbf{b}\times\mathbf{c}\rvert = \lvert \mathbf{b}\times(\mathbf{a}+\mathbf{b})\rvert$ $= \lvert(\mathbf{b}\times\mathbf{a})+(\mathbf{b}\times\mathbf{b})\rvert$ $= \lvert \mathbf{b}\times\mathbf{a}\rvert = \lvert -(\mathbf{a}\times\mathbf{b})\rvert = \lvert \mathbf{a}\times\mathbf{b}\rvert$ (shown) Note that the area of the parallelogram with sides OB and OC is the same as the area of the parallelogram with sides OA and OB (with same base and vertical height)	• Some students wrote $\lvert \mathbf{b}\times\mathbf{c}\rvert$ as the length of a vector perpendicular to both **b** and **c**. This is not an **interpretation** of the geometrical meaning; it is merely translating in words the meaning of modulus of the cross product of **b** and **c**. • In general, $\lvert \mathbf{b}\times(\mathbf{a}+\mathbf{c})\rvert \neq \lvert \mathbf{b}\times\mathbf{a}\rvert + \lvert \mathbf{b}\times\mathbf{c}\rvert$ • $\mathbf{b}\times\mathbf{b} \neq \lvert \mathbf{b}\rvert^2$ but $\mathbf{b}\bullet\mathbf{b} = \lvert \mathbf{b}\rvert^2$
	$\mathbf{a}\times 2\mathbf{b} = \mathbf{d}\times 3\mathbf{a}$ $(\mathbf{a}\times 2\mathbf{b})-(\mathbf{d}\times 3\mathbf{a}) = \mathbf{0}$ $(\mathbf{a}\times 2\mathbf{b})+(3\mathbf{a}\times\mathbf{d}) = \mathbf{0}$ $\mathbf{a}\times(2\mathbf{b}+3\mathbf{d}) = \mathbf{0}$ Vector **a** is parallel to $2\mathbf{b}+3\mathbf{d}$. $\mathbf{a} = k(2\mathbf{b}+3\mathbf{d}), k\in\mathbb{R}$	• There is no division for vectors. Hence, $\mathbf{a}\times 2\mathbf{b} = -\mathbf{a}\times 3\mathbf{d} \not\Rightarrow 2\mathbf{b} = -3\mathbf{d}$ • 3 vectors parallel to a plane (coplanar vectors) $\Rightarrow \mathbf{a} = \lambda\mathbf{b}+\mu\mathbf{d},\ \lambda,\mu\in\mathbb{R}$ However, the above does not necessarily show a linear relationship as it involves two parameters λ and μ. A linear relationship involves only one parameter e.g. k for this question.
		Total Marks: 5

Qn	Suggested Solution	Markers' comments
2(i)	Total area of n rectangles $= \frac{1}{n}\left[(3^1-2)+\left(3^{1+\frac{1}{n}}-2\right)+\left(3^{1+\frac{2}{n}}-2\right)+\ldots\left(3^{1+\frac{n-1}{n}}-2\right)\right]$ $= \frac{1}{n}\left[(3^1-2)+\left(3\left(3^{\frac{1}{n}}\right)-2\right)+\left(3\left(3^{\frac{2}{n}}\right)-2\right)+\ldots\left(3\left(3^{\frac{n-1}{n}}\right)-2\right)\right]$ $= \frac{1}{n}\sum_{r=1}^{n}\left(3\left(3^{\frac{r-1}{n}}\right)-2\right)$ $= -2+\frac{3}{n}\sum_{r=1}^{n}3^{\frac{r-1}{n}}$ (shown) $-2+\frac{3}{n}\sum_{r=1}^{n}3^{\frac{r-1}{n}} = -2+\frac{3}{n}\left(1+3^{\frac{1}{n}}+\ldots+3^{\frac{n-1}{n}}\right)$ $= -2+\frac{3}{n}\left(\frac{\left(3^{\frac{1}{n}}\right)^n-1}{3^{\frac{1}{n}}-1}\right)$ $= -2+\frac{6}{n\left(3^{\frac{1}{n}}-1\right)}$	• As it is a "show" question, the series that show the sum of the area of the rectangles has to be shown clearly. • The last term of the series has to be shown. Ending the series with "…" means that it is an infinite series, i.e., there are infinite number of rectangles. • The width of the rectangles is a constant at $\frac{1}{n}$, while the breadth increases at $\frac{1}{n}$ for every subsequent rectangle. • This is obviously a geometric progression (GP) with common ratio $3^{\frac{1}{n}}$, and so the sum of n terms of a GP is to be used. • Clearly, method of difference is not applicable.
(ii)	$\lim_{n\to\infty} S_n = \int_1^2 3^x-2\,dx$ $= \left[\frac{3^x}{\ln 3}-2x\right]_1^2$ $= \frac{9}{\ln 3}-4-\left(\frac{3}{\ln 3}-2\right)$ $= \frac{6}{\ln 3}-2$	• Students should be able interpret the meaning of $n\to\infty$ in the context of this question. It means that there is an infinite number of rectangles under the curve, this renders the space between the curve and the rectangles negligible, and so $\lim_{n\to\infty} S_n$ is the area under the curve from $x=1$ to $x=2$.
(iii)	Volume $= \pi\int_1^2 (3^x-2)^2-\left(\frac{x^3}{\sqrt{4+x^2}}\right)^2 dx$ $= 38.8056$ $= 38.81$ unit^3 	• $\pi\int_1^2\left(3^x-2-\frac{x^3}{\sqrt{4+x^2}}\right)^2 dx$ is incorrect because it would be the volume form when the curve

	$y=3^x-2-\dfrac{x^3}{\sqrt{4+x^2}}$ is rotated about the x-axis. • To find the required volume, student has to find the volume generated under curve C_1 first, then subtract away the volume generated under curve C_2. • It is important to learn how to key in the expression correctly into the GC to get the correct answer.
	Total Marks: 9

Suggested Solution	Markers' comments
The sequence is <u>increasing</u> from $\dfrac{3}{M}$ to 3.	• Since both n and M are positive integers, $n<M+1 \Rightarrow$ last term $n=M$. • As the sequence is finite with M terms, it's incorrect to say that the sequence "converges to 3". We use the term "converge or diverge" for infinite sequence.
$S_1=u_1=\dfrac{3}{M-1+1}=\dfrac{3}{M}$ $S_2=u_1+u_2=\dfrac{3}{M}+\dfrac{3}{M-2+1}=\dfrac{3}{M}+\dfrac{3}{M-1}$ $S_3=u_1+u_2+u_3=\dfrac{3}{M}+\dfrac{3}{M-1}+\dfrac{3}{M-2}$ $\sum_{n=1}^{M}S_n$ $=S_1+S_2+S_3+\ldots+S_M$ $=(u_1)+(u_1+u_2)+(u_1+u_2+u_3)+\ldots+(u_1+\ldots+u_M)$ $=Mu_1+(M-1)u_2+(M-2)u_3+\ldots+2u_{M-1}+u_M$ $=M\left(\dfrac{3}{M}\right)+(M-1)\left(\dfrac{3}{M-1}\right)+(M-2)\left(\dfrac{3}{M-2}\right)+\ldots+3$ $=3M$	• There is no need to combine the terms into a single fraction. • It's important to list down the terms correctly which will enable us to see the pattern of recurring terms
$S_n=u_1+u_2+\ldots+u_n$ $=\dfrac{3}{M}+\dfrac{3}{M-1}+\ldots+\dfrac{3}{M-n+1}$ $>\dfrac{3}{M}+\dfrac{3}{M}+\ldots+\dfrac{3}{M}=\dfrac{3n}{M}$ (shown)	• Examples : $M-1<M \Rightarrow \dfrac{3}{M-1}>\dfrac{3}{M}$, $M-2<M \Rightarrow \dfrac{3}{M-2}>\dfrac{3}{M}$
	Total Marks: 6

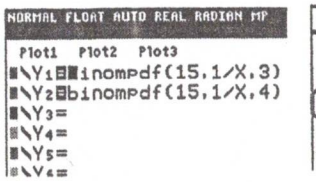
Qn	Suggested Solution	Markers' comments
4(i)	There are 3 stationary points at $x=-3$, 5 and 8. It has a minimum point at $x=-3$, point of inflexion at $x=5$ and maximum point at $x=8$. Note:  	Many students wrote down the coordinates of the stationary points of $y=f(x)$ as (-3,0), (5,0) and (8,0), this is wrong as additional information is needed to know the y-coordinate. This is due to the fact that $f(x)=\int f'(x)\,dx=F(x)+c$.
(ii)	 The gradient is decreasing.	Students will need to know that terms concave upwards / downwards which can be tested in the A-levels.

 ange of x are either $0<x<5$ or $x>7$. **Note that the raph given is a gradient curve which means we are ooking for parts of the curve that decreases as x ncreases.**	
Ie is right because for $-3<x<5$, $f'(x)>0$, is increasing, thus $f(x)>7$	There is a variety of reasons given but the most precise answer is the one given here. Students need to be aware that a positive gradient will imply that as x increases, the change in y will be positive.
$\int_{-3}^{8} xf(x)\,dx = \left[\frac{x^2}{2}f(x)\right]_{-3}^{8} - \int_{-3}^{8}\frac{x^2}{2}f'(x)\,dx$ Since $\frac{x^2}{2}\geq 0$ and $f'(x)\geq 0$ for $-3\leq x\leq 8$, $\therefore \frac{x^2}{2}f'(x)\geq 0$ and $\int_{-3}^{8}\frac{x^2}{2}f'(x)\,dx>0$. Many do not know this concept. $\int_{-3}^{8} xf(x)\,dx$ $=32f(8)-\frac{9}{2}f(-3)-\int_{-3}^{8}\frac{x^2}{2}f'(x)\,dx<32f(8)-\frac{9}{2}f(-3)$ He is wrong to make this statement.	At this point after integration by parts, it should be obvious that you will need to take information from the curve of $y=f'(x)$ in order to answer the qn.
	Total Marks: 8

Qn	Suggested Solution	Markers' comments
5(a)(i)	$(x+y)^2=4e^{xy}$ $2(x+y)\left(1+\frac{dy}{dx}\right)=4e^{xy}\left(x\frac{dy}{dx}+y\right)$ $\frac{dy}{dx}=\frac{2ye^{xy}-y-x}{y+x-2xe^{xy}}$	Note: 1. $e^{xy}\neq e^x e^y=e^{x+y}$ 2. $\ln(4e^{xy})\neq 4xy$. It should be $\ln(4e^{xy})=\ln 4+xy$ 3. Do not attempt to "sq root" the equation as it will result in $(x+y)^2=4e^{xy} \Rightarrow x+y=\pm 2e^{\frac{xy}{2}}$ You will then have to choose the correct equation based on the point A in **(ii)**. Do not overcomplicate your working.
(ii)	When $x=0$, $(0+y)^2=4e^{(0)y}$ $y=2\ (\because y>0)$ When at (0, 2), $\frac{dy}{dx}=\frac{2(2)-2}{2}=1$ Equation of the tangent to the curve at (0, 2) is $y-2=1(x-0)$ $y=x+2$	
(iii)	Substitute $y=x+2$ into $(x+y)^2=4e^{xy}$, $(x+x+2)^2=4e^{x(x+2)}$ $(2x+2)^2=4e^{x^2+2x}$ $(x+1)^2=e^{x^2+2x}$ Using G.C., $x=-2$ or $x=0$ (reject $\because$ it's point A) $\therefore B(-2,0)$	Quite a number of students used the y-coordinate of the intersection between the two curves $y=(x+1)^2$ and $y=e^{x^2+2x}$ as the y-coordinate of B which is wrong. Note that you are solving for the x-coordinate of B in that equation and you will need the equation of the tangent i.e. $y=x+2$ to get the y-coordinate of B.
(b)	Let r and h be the radius and the height of the cylinder respectively. Fixed vol. $p=\pi r^2 h \Rightarrow h=\frac{p}{\pi r^2}$ Surface area, S $=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\frac{p}{\pi r^2}\right)$ $=2\pi r^2+\frac{2p}{r}$ $\frac{dS}{dr}=4\pi r-\frac{2p}{r^2}$	Students need to have a clear picture of which variables are varying. From the qn, it is clear that p is a constant (i.e. it cannot vary). You may vary r but its variation will affect h since they are connected by the volume equation. So for this scenario, you are to vary r such that you can get the minimum surface area.

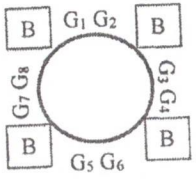
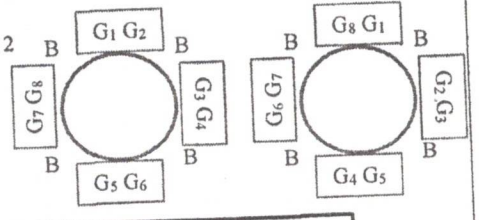
	For min. S, $\frac{dS}{dr}=4\pi r-\frac{2p}{r^2}=0 \Rightarrow r=\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$ $\frac{d^2S}{dr^2}=4\pi+\frac{4p}{r^3}>0$ since r and p are positive. $\therefore$ S is minimum when $r=\left(\frac{p}{2\pi}\right)^{\frac{1}{3}}$ cm.	Note: $\frac{d}{dr}(2\pi rh)\neq 2\pi h$ as h varies with r. For students who use the first derivative test, they should present the working as $\frac{dS}{dr}=4\pi r-\frac{2p}{r^2}=\frac{2(2\pi r^3-p)}{r^2}$ so that the sign of the derivative can be easily observed by the examiner.
		Total Marks: 12

n	Suggested Solution	Markers' comments
(i)	Let X be the random variable denoting the number of questions answered correctly out of 15. $X\sim B\left(15,\frac{1}{n}\right)$ It is always good practice to write down the distribution. $P(X=3)=P(X=4)$ $\binom{15}{3}\left(\frac{1}{n}\right)^3\left(\frac{n-1}{n}\right)^{12}=\binom{15}{4}\left(\frac{1}{n}\right)^4\left(\frac{n-1}{n}\right)^{11}$ $\left(\frac{n-1}{n}\right)=3\left(\frac{1}{n}\right)$ $n=4$ You can refer to MF26 for the pdf for a binomial distribution. Using your GC to check the answer From GC, $n=4$	Common errors: 1. Wrong pdf for binomial. In particular, students tend to miss out the $\binom{15}{r}$ part. 2. Confusing n in the question with n in MF26. 3. Errors in solving the equation.

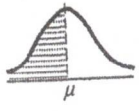


X	Y1	Y2
1	0	0
2	.01389	.04166
3	.12988	.19482
4	.2252	.2252
5	.25014	.1876
6	.23626	.14175
7	.20862	.10431

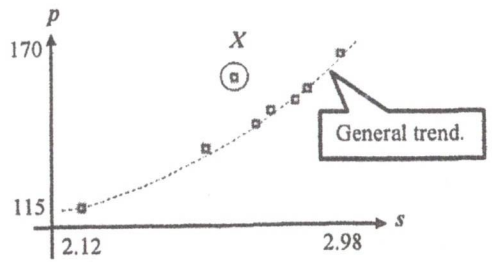
(ii)	$X\sim B\left(15,\frac{1}{4}\right)$ **Method 1** Let T be the score for 15 questions. Then $T=3X-(15-X)=4X-15$ $s=E(T)=4E(X)-15$ $=4\left[15\left(\frac{1}{4}\right)\right]-15=0$ Refer to MF26 for $E(X)$ formula. **Method 2** Let A be the score for 1 question. $E(A)=3\left(\frac{1}{4}\right)-1\left(\frac{3}{4}\right)=0$ $s=E(T)=E(A_1+A_2+\ldots+A_{15})=15E(A)=0$ **Method 3** Let Y be the random variable denoting the number of questions answered incorrectly out of 15. $Y\sim B\left(15,\frac{3}{4}\right)$. $s=3E(X)-E(Y)$ $=3\left[15\left(\frac{1}{4}\right)\right]-\left[15\left(\frac{3}{4}\right)\right]=0$ Note that X and Y are not independent. You need to write Y in terms of X if you want to calculate $Var(3X-Y)$.	Common Errors: 1. Some students assumed that scores must be non-negative. 2. Students who constructed the pdf table on scores for 15 questions often missed out the case $T=-15$. 3. Limitation of GC: Even with the correct pdf, GC can only give s as -2.3×10^{-13} and not the exact answer 0.
(iii)	$P(-4<T<4)=P(-4<4X-15<4)$ $=P(2.75<X<4.75)$ $=P(3\le X\le 4)$ From (i) $P(X=3)=P(X=4)$ $=P(X\le 4)-P(X\le 2)$ or $2P(X=4)$ or $2P(X=3)$ $=0.450$	**Major Concept Error:** Students assumed T to be a normal random variable when T is a discrete random variable.
		Total Marks: 8

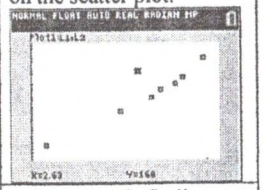
Suggested Solution	Markers' comments
Ways to divide the 12 students with 1 boy in each group $=\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}=2520$ **OR** $\frac{8!}{2!2!2!2!}=2520$ First, put 1 boy in each gp. Next allocate 2 girls to each gp.	There were quite a few students who did this or a variant: $\binom{8}{2}+\binom{6}{2}+\binom{4}{2}+\binom{2}{2}$
P(the 3 members in each group are next to each other) $=\frac{(4-1)!(3!)^4}{(12-1)!}=0.000195$ $(4-1)!=$ ways to sit the 4 gps in a circle $3!=$ ways to arrange the 3 members within each gp	Many students had $(3!)\times 4$ instead of $(3!)^4$
P(every boy is separated from each other by exactly 2 girls) $=\frac{(4-1)!8!}{11!}=\frac{1}{165}$  $(4-1)!=$ ways to sit 4 boys in a circle $8!=$ ways to sit the rest of 8 girls in between the boys **OR** $\frac{(4-1)!(2520)(2!)^4}{11!}=\frac{1}{165}$ 2520 = ways to sit any 2 girls in each of the 4 slots between the boys. Similar to 1st part's concept to allocate 2 girls to each gp which already has 1 boy. Alternative $\frac{(8-1)!(4!)}{11!}\times 2=\frac{1}{165}$  $(8-1)!=$ ways to sit 8 girls in a circle $4!=$ ways to sit the rest of 4 boys in between the girls	Some students gave $\frac{4!8!}{11!}$ which did not consider the circular nature of the problem. Many who used the Alternative 1 did not multiply by 2. There are some who tried to do probability tree method (see alternative 2) but none got it correct due to its complexity and hence caution is advised.

(iii)	The event "No 2 boys are next to each other" is equivalent to "a least 1 girl between every 2 boys". Thus event A is a subset of event B. Now $P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)}{P(B)}\neq P(A)$ as $P(B)<1$. Thus A and B are not independent.	Many students did not explain explicitly with mathematical working. Many students did not indicate $P(B)<1$ but marks are only deducted for those who computed $P(B)$ incorrectly. Note that only $P(B)\neq 1$ is required and hence there is no need to compute $P(B)$.
		Total Marks: 9

Qn	Suggested Solution	Markers' comments
8(i)	$P(\mu<X<16)=0.35$ $P(X<16)-P(X<\mu)=0.35$ $P(X<16)=0.85$ $P\left(Z<\frac{16-\mu}{2}\right)=0.85$ Using GC, $\frac{16-\mu}{2}=1.0364$ $\mu=13.927\approx 14$ (shown)	• An analytical method of standardization and inverse norm is expected. • $P(X<\mu)=0.5$ 
(ii)	Let T be the total time taken by the first two groups of riders. $T=X_1+X_2\sim N(14\times 2, 2^2\times 2)\Rightarrow T\sim N(28,8)$ Required Probability $=P(T<30)$ $=0.76025$ $=0.760$ (3 sf)	• The 12th rider belongs to the third group, and so for the 12th rider to ride before 10.30 am, the first two groups must complete their rides within 30 minutes. • $X_1+X_2\neq 2X$ • $2X$ means twice the time taken by one group of riders. • X_1+X_2 means the sum of the time taken by two groups of riders.

i)	Let Y be the number of long rides out of 4. $Y \sim B(4, P(X \geq 15)) \Rightarrow Y \sim B(4, 0.30854)$ $P(Y \geq 2) = 1 - P(Y \leq 1)$ $= 1 - 0.63661$ $= 0.36339$ $= 0.363$ (shown)	• Sufficient working is necessary since this is a "show" question.
v)	P(none of these rides are consecutive$\mid Y \geq 2$) $= \dfrac{3[P(X \geq 15)]^2 [P(X < 15)]^2}{0.363}$ $= \dfrac{0.13655}{0.363}$ $= 0.376$	• With the words "given that", students should recognize that this is a conditional probability question. • The numerator is the probability for at least 2 long rides and none of these rides are consecutive. This means the 3 possible scenarios are LSLS, SLSL, LSSL (L: for long ride; S: for short ride).
		Total Marks: 9

n	Suggested Solution	Markers' comments
i)	 Since there is no controlled variable, both p on s as well as s on p scatter plots are accepted.	Common Errors: 1. Wrong labelling of axes. 2. Careless plotting of points: the points are not always accurately plotted, leading to wrong point X identified. Popular <u>wrong</u> points identified are: the first point, the last point or the 4th point.

(ii)	Omitting X, $r = 0.995$ [between p and s^5] $r = 0.976$ [between $\ln p$ and s] Since the r-value between p and s^5 is closer to 1 than that between $\ln p$ and s (indicating a stronger linear relationship between p and s^5), $p = a + bs^5$ is the better model. Note that $\ln p = c + ds \Rightarrow p = e^c e^{ds}$ curve has the same shape as $p = a + bs^5$. Thus we are not able to decide the better model purely from the scatter plot.	Students tend to forget to delete the point X, or they deleted a neighbouring point. You can press 'trace' button to track the points on the scatter plot. 
(iii)	From GC, $p = 101.449 + 0.279507s^5$ When $s = 2.63$, $p = 136.618 = 137$ (3 sf) Two reasons: 1. The r-value (0.995) is very close to 1, indicating a strong fit of model to the data. 2. $s = 2.63$ lies within data range $2.12 \leq s \leq 2.98$, thus interpolation is used. Interpolation is based on the <u>given</u> value (in this case $s = 2.63$) lying within data range, not the estimated one.	• Instead of finding an estimate for p when $s = 2.63$, some students proceeded to find the correct data value for p when $s = 2.63$. Realise that there should not be a follow up question on reasons for the data value to be reliable. • For those who have chosen X to be the first (or last) point, you should realise by now that the estimate is unreliable because it is an extrapolation.
(iv)	High correlation **<u>does not necessarily</u>** imply causation. There could be other factors (e.g. strong economy, increase in population) which can lead to higher property price index. So the higher property price index need not be caused by higher stock index.	Note that both statements need to be stated to give a complete answer in context of the question, which some students were not able to do during the exam.
		Total Marks: 10

n	Solution	Markers' comments
(i)	$\sum \mathrm{P}(X=x)=1$ $\sum_{x=1}^{n}\frac{ax}{n(n+1)}=1$ $\frac{a}{n(n+1)}\sum_{x=1}^{n}x=1$ $\frac{a}{n(n+1)}\left(\frac{n(n+1)}{2}\right)=1$ $a=2$ (shown) Alternative When $n=1, \mathrm{P}(X=1)=\frac{a}{1(2)}=1 \therefore a=2$	• For any discrete random variable X, the sum of probability values for all possible values of x is 1. • Note that in this case, a and n are constants so they can be factored out from the sum in the first method. • The resultant sum is an A.P so apply the A.P formula. (In general, note that it is possible for pure math knowledge to apply in statistics questions)
)	$\mathrm{E}(X)=\sum_{x\in S}x\mathrm{P}(X=x)$ $=\frac{a}{n(n+1)}\sum_{x=1}x^2$ $=\frac{2}{n(n+1)}\left(1^2+2^2+3^2+\ldots+n^2\right)$ $=\frac{2}{n(n+1)}\sum_{r=1}^{n}r^2$ $=\frac{2}{n(n+1)}\left[\frac{n}{6}(n+1)(2n+1)\right]$ $=\frac{1}{3}(2n+1)$	• This part is generally well done.
i)	For this game, the tokens are picked without replacement thus the two tokens picked by the players will be different. For any two possible tokens picked, say a and b, there are two equally possible outcomes: P_1 picks a and P_2 picks b or vice versa, out of which only one where P_2 will lose. Thus the probability of P_2 losing is 0.5. Alternative	• Some elaboration is required here, beyond stating that there will be no draw and rephrasing the statement that both P_1 and P_2 are equally likely to lose.

	P(P_2 loses a game) or P(P_1 wins a game) $=\left(\frac{1}{n+1}\right)\left(\frac{n}{n}\right)+\left(\frac{1}{n+1}\right)\left(\frac{n-1}{n}\right)+\left(\frac{1}{n+1}\right)\left(\frac{n-2}{n}\right)+\ldots+\left(\frac{1}{n+1}\right)\left(\frac{1}{n}\right)$ Prob P_1 picks the number "$n+1$" Prob P_1 picks the number "n" Prob P_1 picks the number "2" Prob P_2 picks a number smaller than "2" Prob P_2 picks a number smaller than "$n+1$" Prob P_2 picks a number smaller than "n" $=\frac{1}{n(n+1)}(1+2+3+\ldots+n)$ $=\frac{1}{n(n+1)}\left(\frac{n}{2}(1+n)\right)$ $=\frac{1}{2}$	
(iii)	Required probability $=\mathrm{P}(P_2 \text{ loses } \$(m+1) \mid P_2 \text{ loses})$ $=\frac{\mathrm{P}(P_2 \text{ loses } \$(m+1))}{\mathrm{P}(P_2 \text{ loses})}$ $=\frac{\mathrm{P}(P_1 \text{ picks "}m+1\text{"})\times\mathrm{P}(P_2 \text{ picks a no. smaller than "}m+1\text{"})}{\mathrm{P}(P_2 \text{ loses})}$ $=\frac{\left(\frac{1}{n+1}\right)\left(\frac{m}{n}\right)}{\frac{1}{2}}$ $=\frac{2m}{n(n+1)}$	• Note that the required probability is conditional on the event that P_2 loses. • To calculate the denominator i.e. P(P_2 loses \$$m$+1), note that P_1 must draw the tile $m+1$ and then P_2 must draw a tile smaller than m+1. (For convenience's sake, you may assume P_1 draws first but the answer is still the same even if P_2 draws first)

)	Let L be the random variable denoting the amount lost for a game if he loses a game. $L = X + 1$ $\mathrm{E}(L)$ $= \mathrm{E}(X+1)$ $= \left(\frac{1}{3}(2n+1)\right) + 1$ $= \frac{1}{3}(2n+4)$	• First define the random variable L, before trying to link it to X. • Note that $0 \le L \le n+1$ and $\mathrm{P}(L = m+1)$ $= \mathrm{P}(X = m)$ • Thus $L - 1 = X$.
		Total Marks : 12

n	Suggested Solution	Markers' comments
(i)	Unbiased estimate of population mean, $\bar{x} = \frac{\sum x}{n} = \frac{5415}{120} = 45.125$ (exact) Unbiased estimate of population variance, $s^2 = \frac{1}{119}\left[351500 - \frac{(5415)^2}{120}\right] = 900.40 = 900$ (3 s.f.)	• Remember to use the exact or 5 s.f. answer for subsequent working.
i)	An estimate is unbiased when the **expected value of the estimator T** used to obtain the estimate is equal to the **value of the population parameter θ**, i.e. $\mathrm{E}(T) = \theta$.	• See Tut19 Q5(ii)
ii)	Let X denote the speed a randomly chosen car (in km/h) with population mean μ. To test $\mathrm{H}_0 : \mu = 50$ vs $\mathrm{H}_1 : \mu < 50$ Conduct 1-tail test at 3% significance level. Under H_0, $\bar{X} \sim \mathrm{N}\left(50, \frac{900.40}{120}\right)$. Use the 5 s.f figure for better accuracy Using a z-test, $p\text{-value} = \mathrm{P}(\bar{X} \le 45.125) = 0.0376$ (3 s.f.) Since p-value > 0.03, we do not reject H_0 and conclude that there is insufficient evidence at 3% significance level that the <u>mean speed is less than 50 km/h</u>. <u>Hence WRP is not needed.</u>	• Important to define the random variable X. • X normal $\Rightarrow \bar{X}$ normal, no need to use CLT. • Important to answer the question at the end i.e. <u>"WRP is not needed".</u>

(iv)	Unbiased estimate of population variance, $s^2 = \frac{80}{79}[1100] = 1113.9 = 1110$ (3 s.f.) To test $\mathrm{H}_0 : \mu = \mu_0$ vs $\mathrm{H}_1 : \mu > \mu_0$ Conduct 1-tail test at 8% significance level. Under H_0, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{1113.9}{80}\right)$ If mean speed exceeds $\mu_0 \Rightarrow \mathrm{H}_0$ is rejected, $\Rightarrow p\text{-value} = \mathrm{P}(\bar{X} \ge 60) \le 0.08$ $\mathrm{P}\left(Z \ge \frac{60-\mu_0}{\sqrt{\frac{1113.9}{80}}}\right) \le 0.08$ $\frac{60-\mu_0}{\sqrt{\frac{1113.9}{80}}} > 1.4051$ $\mu_0 < 54.757$ $\therefore$ maximum $\mu_0 = 54.7$	• The sample variance = 1100. We need to find the unbiased estimate by using the formula : $s^2 = \frac{n}{n-1}$(sample var) • Qn requires the use of an algebraic method to find max μ_0 as shown on the left. A GC method is not acceptable.
(v)	Assumption : 1. The speeds of the second sample of cars comes from a random sample, **OR** 2. The speeds of the second sample of cars are independent	
		Total Marks: 12