H1

TAMPINES JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

MATHEMATICS **8865/01**

Paper 1

Monday, 10 September 2018

3 hours

Additional Materials: Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer all the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Find algebraically the range of values of k for which

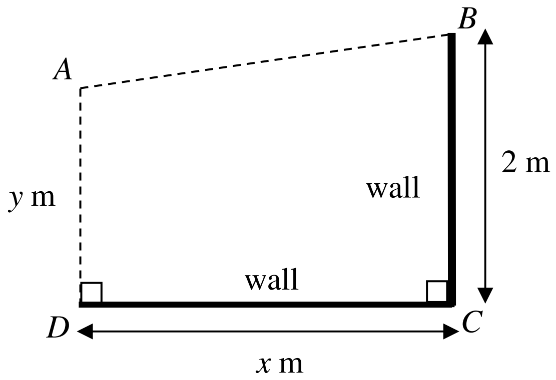
$$3kx^2 - 6x + k > 0$$

for all real values of x. [4]

2 **(i)** Differentiate $\dfrac{1}{3x^2+4}$ with respect to x. [2]

(ii) Find $\int \left(3-\sqrt{x}\right)^2 \, dx$. [3]

3



The diagram shows a garden *ABCD* in the shape of a trapezium next to the walls *BC* and *CD*. The wall *BC* is 2 m long and the wall *CD* is x m long. The broken lines *AB* and *AD* represent fences. The fence *AD* is y m long and is parallel to *BC*. The total length of the fences is 5 m.

(i) Show that the area, A m^2, of the garden *ABCD* is given by

$$A = \frac{1}{12}\left(33x - x^3\right).$$ [3]

(ii) Use differentiation to find the maximum value of A. Justify that this is the maximum value. [4]

4 The curve C has equation $y = e^{2x-1} - 3$.

(i) Sketch the graph of C, stating the exact coordinates of any points of intersection with the axes and the equations of any asymptotes. [3]

(ii) Without using a calculator, find the equation of the tangent to C at the point where $x = \frac{1}{2}$, giving your answer in the form $y = mx + c$, where m and c are constants. [4]

(iii) Find the exact area of the region bounded by C, the x-axis and the line $x = 2$. [4]

5 A student decided to track his daily expenditure, \$$S$, for t days, $1 \le t \le 20$. He spent \$12 on Day 1. Every day, he spent more money than the previous day up till Day 10. On Day 10, he spent \$30 and realised that he had used up too much of his budget. He decided to cut his expenditure and spent less money than the previous day for the remaining days. He noticed that his expenditure can be modelled using a quadratic equation.

(i) Show that the expenditure is modelled by

$$S = -\frac{2}{9}t^2 + \frac{40}{9}t + \frac{70}{9}.$$ [4]

(ii) Find the student's expenditure on Day 20. [1]

(iii) Sketch the graph of S against t, for $1 \le t \le 20$, stating the coordinates of the end points and turning point. [2]

(iv) Would this be a good equation to use to estimate his daily expenditure in the long run? Justify your answer. [1]

The student's mother recommended that he could set a daily budget of \$20 instead.

(v) By adding a suitable line to your graph in **(iii)**, find the range of Days for which the student would have exceeded this daily budget. [3]

To improve his financial planning, the student should set a new daily budget which is the minimum amount he does not exceed 75% of the time based on his previous record.

(vi) How much should this new daily budget be? [2]

Section B: Probability and Statistics [60 marks]

6 A bakery produces two types of cookie. One type of cookie contains nuts, and the other type contains no nuts. There is a constant probability that a cookie contains nuts. The cookies are sold in packs of 6. Each pack has a random selection of cookies. For these packs, the mean number of cookies containing nuts is 0.8.

(i) Find the probability that a pack chosen at random has at least 2 cookies containing nuts. [3]

A customer buys 10 packs of cookies for a party.

(ii) Find the probability that less than 4 of these packs have at least 2 cookies containing nuts. [2]

7 A group of 7 boys and 5 girls are standing in a queue. Find the number of different possible arrangements if

(i) the girls must stand together, [2]

(ii) all the girls must be separated. [2]

Find the probability that there are at most 2 boys in the front half of the queue. [3]

8 The year, x, and the mean amount spent on credit cards per household in Singapore in the 1st Quarter, y thousand dollars, are given in the following table.

x	2000	2003	2006	2009	2011	2013	2015	2017
y	2.01	3.00	3.63	5.23	6.73	8.73	9.95	10.3

(i) Give a sketch of the scatter diagram for the data, as shown on your calculator. [2]

(ii) Find the product moment correlation coefficient. [1]

(iii) Find the equation of the regression line of y on x, in the form $y = mx + c$, giving the values of m and c correct to 3 significant figures. Sketch this line on your scatter diagram. [2]

(iv) Calculate an estimate of the mean amount spent on credit cards per household in Singapore in the 1st Quarter of 1998. Comment on the reliability of your estimate. [2]

(v) Without calculating the estimate, state two reasons why you would expect the estimate of the mean amount spent on credit cards per household in Singapore in the 1st Quarter of 2016 to be reliable. [2]

9 A riding qualification involves two separate parts, theory and practical. To succeed in the riding qualification, a student must first pass theory followed by practical. Students who fail theory at the first attempt always make a second attempt, while students who pass theory at the first attempt cannot make a second attempt. Students are allowed at most two attempts at theory but only one attempt at practical.

A is the event that the student passes theory at the first attempt,
B is the event that the student passes practical,
C is the event that the student passes theory at the second attempt.

It is given that $\mathrm{P}(A) = 0.2$, $\mathrm{P}(B) = 0.4$, $\mathrm{P}(C \mid A') = 0.7$ and $\mathrm{P}(A \cup B) = 0.52$.

(i) Determine whether the events *A* and *B* are independent. [2]

(ii) Explain, in the context of the question what is meant by $\mathrm{P}(C \mid A)$, and find its value. [2]

(iii) Draw a tree diagram to represent the information above. [2]

(iv) Find the probability that a student chosen at random will succeed in the riding qualification. [2]

There are *n* students who take the riding qualification.

(v) Find the least value of *n* given that the probability that none of the students will succeed in the riding qualification is less than 0.1. [4]

10 The masses, in kg, of two types of oranges, A and B, sold by a supermarket have independent normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard deviation
Type A	0.26	σ
Type B	0.16	0.02

It is found that 40% of oranges of type A have a mass less than 0.25 kg.

(i) Show that the standard deviation of the distribution of the mass of oranges of type A is 0.04 kg, correct to 2 decimal places. [2]

(ii) Find the probability that two randomly chosen oranges of type A each have a mass of more than 0.25 kg, giving your answer correct to 2 decimal places. [2]

(iii) Without any calculation, explain why the probability that the total mass of two randomly chosen oranges of type A is more than 0.5 kg is greater than your answer to part **(ii)**. [1]

4 oranges of type A and 3 oranges of type B are chosen at random.

(iv) Find the probability that the total mass of 4 oranges of type A is at least 0.6 kg more than the total mass of 3 oranges of type B. [4]

Oranges of type A cost \$4.50 per kg and oranges of type B cost \$5 per kg.

(v) Find the probability that the total cost of 4 oranges of type A and 3 oranges of type B is between \$6.50 and \$7.50. State the mean and variance of the distribution that you use.

[4]

11 Intensity of light is measured in lumens. A light bulb manufacturing company claims that the mean intensity of light from its standard 60 watt light bulbs is at least 800 lumens. A random sample of 50 standard 60 watt light bulbs is checked and the intensity of light from the light bulbs, x lumens, are summarised by

$$\sum(x-800)=-300, \qquad \sum(x-800)^2=34924.$$

(i) Calculate unbiased estimates of the population mean and variance. [3]

(ii) Test, at the 5% significance level, whether the company's claim is valid. [5]

(iii) State the meaning of the p-value in context. [1]

(iv) State, giving a reason, whether it is necessary to assume a normal distribution for this test to be valid. [1]

The manufacturing company claims that the mean intensity of light from its energy-efficient 15 watt light bulbs is 850 lumens. It is known that the standard deviation of the intensity of light from energy-efficient 15 watt light bulbs is 10 lumens. A consumer organisation decides to check the manufacturer's claim and measures the intensities of a random sample of 20 energy-efficient 15 watt light bulbs. Using a 5% significance level, the consumer organisation finds that the manufacturer overestimated the intensity of light from its energy-efficient 15 watt light bulbs.

(v) Find the set of values within which the mean intensity of light from the random sample of 20 energy-efficient light bulbs must lie. [4]

End of Paper

2018 JC2 Preliminary Examination
H1 Mathematics
Solution

1 **[4m]**	$3kx^2-6x+k>0$ $3k>0$ and Discriminant <0 Discriminant $=(-6)^2-4(3k)(k)$ $=36-12k^2$ $36-12k^2<0$ $3-k^2<0$ $(\sqrt{3}-k)(\sqrt{3}+k)<0$ $k<-\sqrt{3}$ or $k>\sqrt{3}$ Since $3k>0$, $k>\sqrt{3}$	

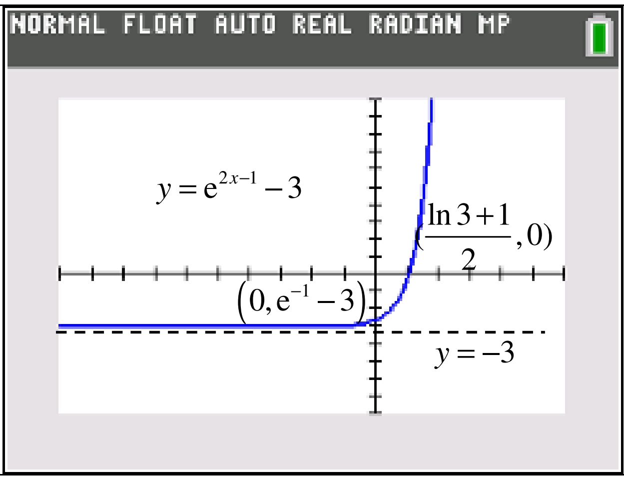
2(i) **[5m]**	$y=\frac{1}{3x^2+4}=\left(3x^2+4\right)^{-1}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(3x^2+4\right)^{-2}(6x)$ $=-\frac{6x}{\left(3x^2+4\right)^2}$	
(ii)	$\int\left(3-\sqrt{x}\right)^2\mathrm{d}x$ $=\int\left(9-6\sqrt{x}+x\right)\mathrm{d}x$ $=9x-\frac{6x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^2}{2}+c$ $=9x-4x^{\frac{3}{2}}+\frac{x^2}{2}+c$	

3(i) **[7m]**	$A=\frac{1}{2}(y+2)x \quad ----(1)$ Using pythagoras theorem, $x^2=(5-y)^2-(2-y)^2$ $x^2=\left(25-10y+y^2\right)-\left(4-4y+y^2\right)$ $x^2=21-6y$ $y=\frac{21-x^2}{6} \quad ----(2)$	

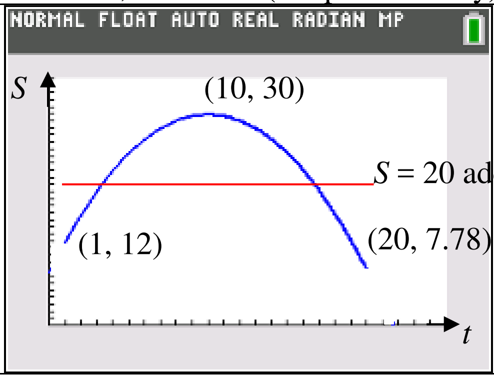
	Substitute (2) into (1) $A=\frac{1}{2}\left(\frac{21-x^2}{6}+2\right)x$ $=\frac{1}{2}\left(\frac{33-x^2}{6}\right)x$ $=\frac{1}{12}\left(33x-x^3\right)$ (shown)	
(ii)	$\frac{dA}{dx}=\frac{1}{12}\left(33-3x^2\right)$ When $\frac{dA}{dx}=0,$ $\frac{1}{12}\left(33-3x^2\right)=0$ $33-3x^2=0$ $x^2=11$ $x=\pm\sqrt{11}$ Since $x>0$, $x=\sqrt{11}=3.3166=3.32$ m (to 3 s.f) To show A is maximum: Using first derivative test, (see table below) Or Using second derivative test, $\frac{d^2A}{dx^2}=\frac{1}{12}(-6x)=-\frac{x}{2}$ When $x=\sqrt{11}$, $\frac{d^2A}{dx^2}=-\frac{\sqrt{11}}{2}<0$ Therefore A is maximum when $x=\sqrt{11}$. Maximum area $=\frac{1}{12}\left(33\left(\sqrt{11}\right)-\left(\sqrt{11}\right)^3\right)=6.08\,\text{m}^2$	

First derivative test table:

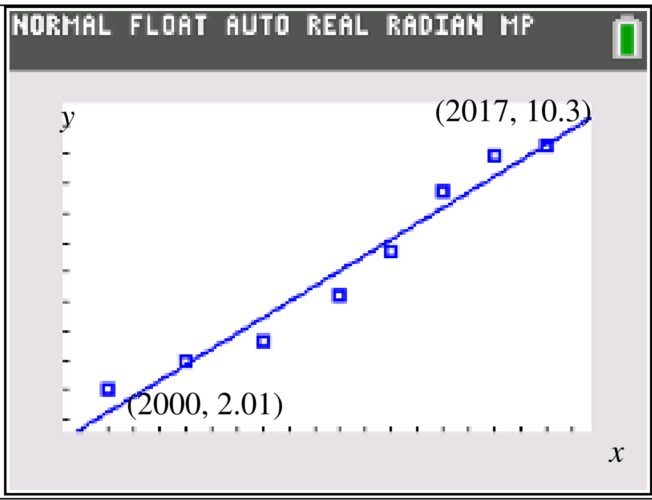
x	$\sqrt{11}^-$	$\sqrt{11}$	$\sqrt{11}^+$
Sign of $\frac{dA}{dx}$	+	0	-
	/	–	\

4(i) **[11m]**	 	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2x-1}$ When $x = \frac{1}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\mathrm{e}^{2\left(\frac{1}{2}\right)-1} = 2$, $y = \mathrm{e}^{2\left(\frac{1}{2}\right)-1} - 3 = -2$ Equation of tangent at P: $y-(-2) = 2(x-\frac{1}{2})$ $y = 2x-3$	
(iii)	$\int_{\frac{\ln 3+1}{2}}^{2} \mathrm{e}^{2x-1} - 3 \, \mathrm{d}x$ $= \left[\frac{\mathrm{e}^{2x-1}}{2} - 3x\right]_{\frac{\ln 3+1}{2}}^{2}$ $= \left[\frac{\mathrm{e}^{2(2)-1}}{2} - 3(2)\right] - \left[\frac{\mathrm{e}^{2\left(\frac{\ln 3+1}{2}\right)-1}}{2} - 3\left(\frac{\ln 3+1}{2}\right)\right]$ $= \frac{\mathrm{e}^3}{2} - 6 - \frac{3}{2} + \frac{3\ln 3}{2} + \frac{3}{2}$ $= \frac{\mathrm{e}^3}{2} + \frac{3\ln 3}{2} - 6$	

5(i) **[13m]**	Let $S = at^2 + bt + c$ \$12 on Day 1: $12 = a(1)^2 + b(1) + c$	

	\$30 on Day 10 is maximum: $30 = a(10)^2 + b(10) + c$ $\frac{dS}{dt} = 2at + b$ $0 = 2a(10) + b$ Using GC, $a = -\frac{2}{9}, b = \frac{40}{9}, c = \frac{70}{9}$	
(ii)	At $t = 20$, $S = \$7.78$ (2 d.p. for money)	
(iii)		
(iv)	No, as his expenditure would eventually become negative which is impossible.	
(v)	Add horizontal line $S = 20$ Intersection points at 3.29 and 16.7 He exceeded \$20 from Day 4 to Day 16.	
(v)	75% of 20 days = 15 days At t = 7, S = 28. The new daily budget should be \$28.	

6(i) **[5m]**	Let X be the number of cookies containing nuts out of 6. $X \sim \mathrm{B}(6, p)$ Since $\mathrm{E}(X) = 0.8$ $6p = 0.8$ $p = \frac{2}{15}$ $\mathrm{P}(X \geq 2)$ $= 1 - \mathrm{P}(X \leq 1)$ $= 0.18509$ (to 5 s.f) $= 0.185$ (to 3 s.f)	
(ii)	Let Y be the number of packs with at least 2 cookies containing nuts out of 10. $Y \sim \mathrm{B}(10, 0.18509)$	

	$\mathrm{P}(Y<4)=\mathrm{P}(Y\le 3)$ $=0.904$ (3sf)	
7(i) **[7m]**	Number of arrangements = $8! \times 5!$ = 4838400	
(ii)	Number of arrangements = $7! \times \binom{8}{5} \times 5!$ = 33868800	
	Probability = $\dfrac{\text{No. ways 1 boy + No. ways 2 boys}}{\text{Total no. ways}}$ $=\dfrac{\binom{7}{1}\times 6!+\binom{7}{2}\times\binom{5}{4}\times 6!}{\binom{12}{6}\times 6!}$ $=\dfrac{4}{33}$	
8(i) **[9m]**		
(ii)	By GC, product moment correlation coefficient r = 0.980 (3sf)	
(iii)	By GC, $y=0.53403x-1066.8$ (5sf) $y=0.534x-1070$ (3sf)	
(iv)	$y=0.53403(1998)-1066.8$ (5sf) $=0.192$ thousand dollars (3sf) The estimate is not reliable as 1998 lies outside the data range. Extrapolation is not a good practice.	
(v)	The correlation coefficient is very close to 1, indicating a strong positive linear relationship. 2016 also lies within the data range and interpolation is a good practice. Hence, the estimate is reliable.	

9 (i) $P(A \cap B) = 0.2 + 0.4 - 0.52$

$= 0.08$

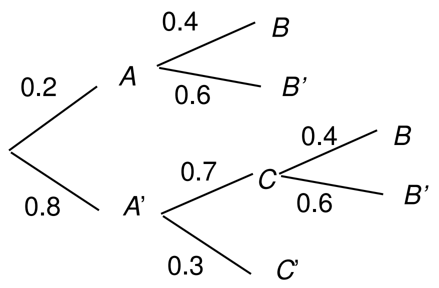
$= 0.2 \times 0.4$

$= P(A) \times P(B)$

Thus A and B are independent.

(ii) $P(C \mid A)$ is the probability of a student passing theory at second attempt if he passed theory at first attempt. Since this is not possible as C and A are mutually exclusive, $P(C \mid A) = 0$

(iii)



(iv) Probability $= (0.2)(0.4) + (0.8)(0.7)(0.4)$

$= 0.304$

(v) Let Y be the number of students who succeed out of n.

$Y \sim B(n, 0.304)$

$P(Y = 0) < 0.1$

Using GC,

n	$P(Y = 0)$
6	0.1137
7	0.0791

Least $n = 7$

10(i)
[13m]

Let A be the mass of oranges of type A, in kg.

$A \sim N(0.26, \sigma^2)$

$P(A < 0.25) = 0.4$

$$P\left(Z < \frac{0.25 - 0.26}{\sigma}\right) = 0.4$$

$$P\left(Z < \frac{-0.01}{\sigma}\right) = 0.4$$

$$\frac{-0.01}{\sigma} = -0.25335$$

$\sigma = 0.0395 = 0.04$ (shown)

(ii) Required probability

	$= \mathrm{P}(A > 0.25) \times \mathrm{P}(A > 0.25)$ $= (0.6)^2$ $= 0.36 \text{ (to 2 d.p)}$	
(iii)	Part **(iii)** includes more cases in addition to the case in part **(ii)**. For example, the mass of one orange of type A is less than 0.25 kg but the mass of the other orange of type A is more than 0.25 kg such that the total mass of two randomly chosen oranges of type A is more than 0.5 kg.	
(iv)	Let *A* and *B* be the mass of oranges of type A and type B, in kg, respectively $A \sim \mathrm{N}(0.26, 0.04^2)$ $B \sim \mathrm{N}(0.16, 0.02^2)$ $(A_1 + A_2 + A_3 + A_4) - (B_1 + B_2 + B_3) \sim \mathrm{N}(4(0.26) - 3(0.16), 4(0.04^2) + 3(0.02^2))$ $(A_1 + A_2 + A_3 + A_4) - (B_1 + B_2 + B_3) \sim \mathrm{N}(0.56, 0.0076)$ $\mathrm{P}((A_1 + A_2 + A_3 + A_4) \geq (B_1 + B_2 + B_3) + 0.6)$ $= \mathrm{P}((A_1 + A_2 + A_3 + A_4) - (B_1 + B_2 + B_3) \geq 0.6)$ = 0.323 (to 3 s.f)	
(v)	$4.5(A_1 + A_2 + A_3 + A_4) + 5(B_1 + B_2 + B_3) \sim \mathrm{N}(7.08, 0.1596)$ $\mathrm{P}(6.50 < 4.5(A_1 + A_2 + A_3 + A_4) + 5(B_1 + B_2 + B_3) < 7.50)$ = 0.780 (to 3 s.f)	

11(i) **[14m]**	Unbiased estimate of population mean $\bar{x} = \frac{-300}{50} + 800 = 794$ Unbiased estimate of population variance $s^2 = \frac{1}{49}\left(34924 - \frac{(-300)^2}{50}\right)$ $= 676$	
(ii)	Let μ be the population mean intensity of light from the standard 60 watt light bulb, in lumens Let *X* be the intensity of light from the standard 60 watt light bulb, in lumens To test $\mathrm{H}_0 : \mu = 800$ against $\mathrm{H}_1 : \mu < 800$ at 5% level of significance. Since *n* = 50 is large, by Central Limit Theorem, $\overline{X} \sim \mathrm{N}\left(800, \frac{676}{50}\right)$ approximately under H_0.	

	Test statistic: $Z=\dfrac{\overline{X}-800}{\sqrt{\dfrac{676}{50}}}\sim \mathrm{N}(0,1)$ approximately under H_0 $z_{test}=\dfrac{794-800}{\sqrt{\dfrac{676}{50}}}=-1.63$ (to 3 s.f) Using G.C., $p=0.051362=0.0514$, $z_{test}=-1.63$, $\bar{x}=794$, $n=50$ Since $p=0.0514>0.05$, we do not reject H_0 and conclude that there is insufficient evidence, at the 5% level of significance, that the mean intensity of light from the standard 60 watt light bulb is less than 800 lumens. Hence the company's claim is valid.	
(iii)	The smallest level of significance such that the company's claim that the mean intensity of light from the standard 60 watt light bulbs is at least 800 lumens is rejected is 5.14%.	
(iv)	It is not necessary to assume normal distribution. Since *n* = 50 is large, by Central Limit Theorem, the mean intensity of light from the standard 60 watt light bulbs is approximately normally distributed.	
(v)	Let μ be the population mean intensity of light from the energy-efficient 15 watt light bulbs Let *Y* be the intensity of light from the energy-efficient 15 watt light bulbs , in lumens To test $\mathrm{H}_0: \mu=850$ against $\mathrm{H}_1: \mu<850$ at 5% level of significance. Since *n* = 20 is large, by Central Limit Theorem, $\overline{Y}\sim \mathrm{N}\left(850, \dfrac{10^2}{20}\right)$ approximately under H_0. Test statistic: $Z=\dfrac{\overline{Y}-850}{\sqrt{\dfrac{10^2}{20}}}\sim \mathrm{N}(0,1)$ under H_0 $z_{\text{test}}=\dfrac{\bar{y}-850}{\sqrt{\dfrac{10^2}{20}}}=\dfrac{\bar{y}-850}{\sqrt{5}}$ Since the manufacturer overestimated the intensity of light from its energy-efficient 15 watt light bulbs, we reject H_0 To reject H_0, $z_{\text{test}}<z_{\text{crit}}$	

	$\dfrac{\overline{y}-850}{\sqrt{5}} < -1.6449$ $\overline{y}-850 < -3.6780$ $\overline{y} < 846.322$ $\overline{y} \le 846$ (to 3 s.f) Set of values = $\{\overline{y} \in \mathbb{R} : \overline{y} \le 846\}$	